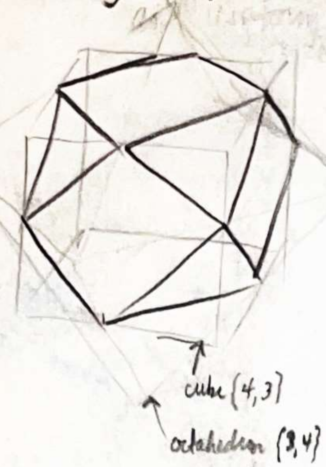
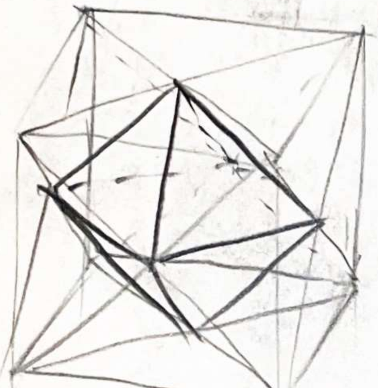
Saturday, Mar. 22, 1958

31

Summary of quasi-regular polyhedra



$\begin{Bmatrix}3\\4\end{Bmatrix} \left(= \begin{Bmatrix}4\\3\end{Bmatrix}\right)$



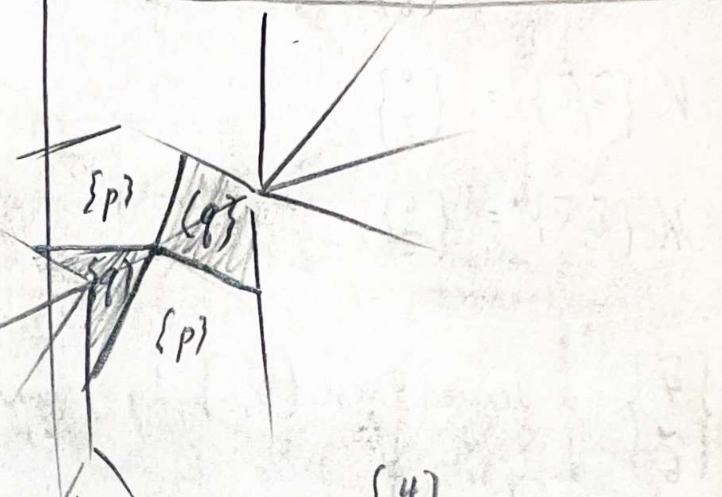
2 reciprocal $\{3,3\}$'s and $\begin{Bmatrix}3\\3\end{Bmatrix}$ $(= \{3,4\})$.

(Coxeter, RP, p. 17) [paraphrased)

Consider 2 regular polyhedra $\{p,q\}$ and $\{q,p\}$ which are reciprocal.

The polyhedron $\begin{Bmatrix}p\\q\end{Bmatrix} \left(= \begin{Bmatrix}q\\p\end{Bmatrix}\right)$ constructed with vertices at the mid-edge points of either $\{p,q\}$ or $\{q,p\}$ has faces $\{q\}$ and $\{p\}$ which are the vertex figures of $\{p,q\}$ and $\{q,p\}$ respectively.

There are 4 edges at each vertex



The reciprocation operator (even in the case of infinite regular polyhedra) interchanges the numbers (e.g., per unit lattice cell) N_0 and N_2 of vertices and faces, leaving the number of edges unchanged.

$\mathcal{M}\{\tilde{4},\tilde{6}\} = \begin{Bmatrix}4\\ \tilde{6}\end{Bmatrix}$

$\mathcal{M}\{\tilde{6},\tilde{4}\} = \begin{Bmatrix}6\\ \tilde{4}\end{Bmatrix}$

Thus, in contrast to the case of the [reciprocal] cube and octahedron $[\{4,3\}$ & $\{3,4\}]$, where $\mathcal{M}\{4,3\} = \mathcal{M}\{3,4\}$, here we have

$$\mathcal{M}\{\tilde{4},\tilde{6}\} = \left[\mathcal{M}\{\tilde{6},\tilde{4}\}\right]'$$

i.e. $\begin{Bmatrix}4\\ \tilde{6}\end{Bmatrix} = \begin{Bmatrix}\tilde{4}\\ 6\end{Bmatrix}'$

(Transpose: move $\sim$ from p to q or vice versa)

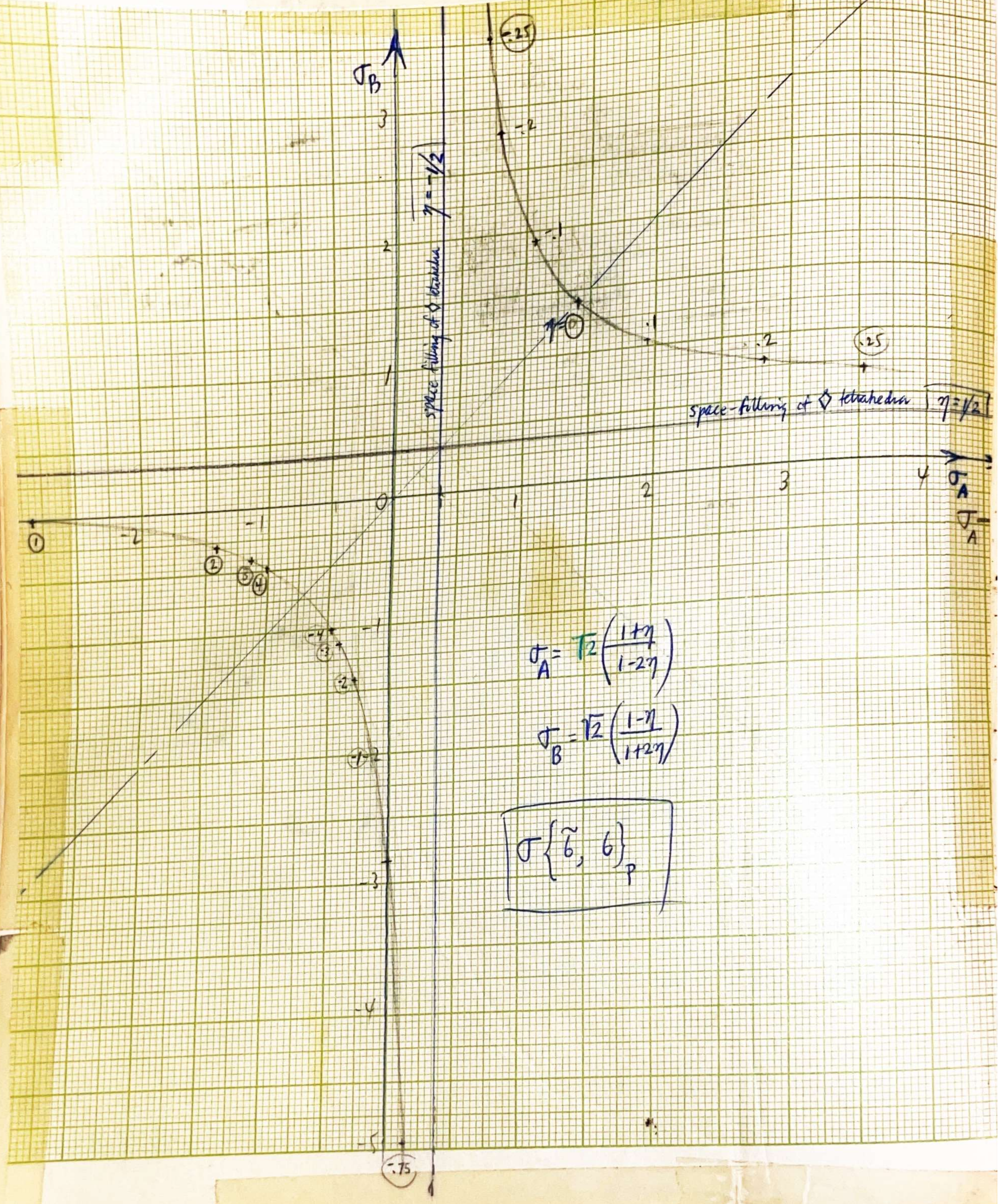
$\mathcal{N}\{\tilde{6},\tilde{4}\} = \begin{Bmatrix}3\\ \tilde{4}\end{Bmatrix}$ $\quad$ $\mathcal{M}\{\tilde{6},\tilde{3}\} = \begin{Bmatrix}6\\ \tilde{6}\end{Bmatrix}$

$\mathcal{N}\{\tilde{4},\tilde{6}\} = \{\tilde{6},\tilde{6}\}$

The operator $\mathcal{N}$ has complicated effects, i.e., sometimes it is inapplicable (e.g., $\{3,6\}$), sometimes it produces a regular figure ($\{\tilde{6},\tilde{6}\}$ from $\{\tilde{4},\tilde{6}\}$), sometimes a quasi-regular figure.

or $\begin{Bmatrix}4\\ \tilde{6}\end{Bmatrix} = T\begin{Bmatrix}\tilde{4}\\ 6\end{Bmatrix}$

(transpose)



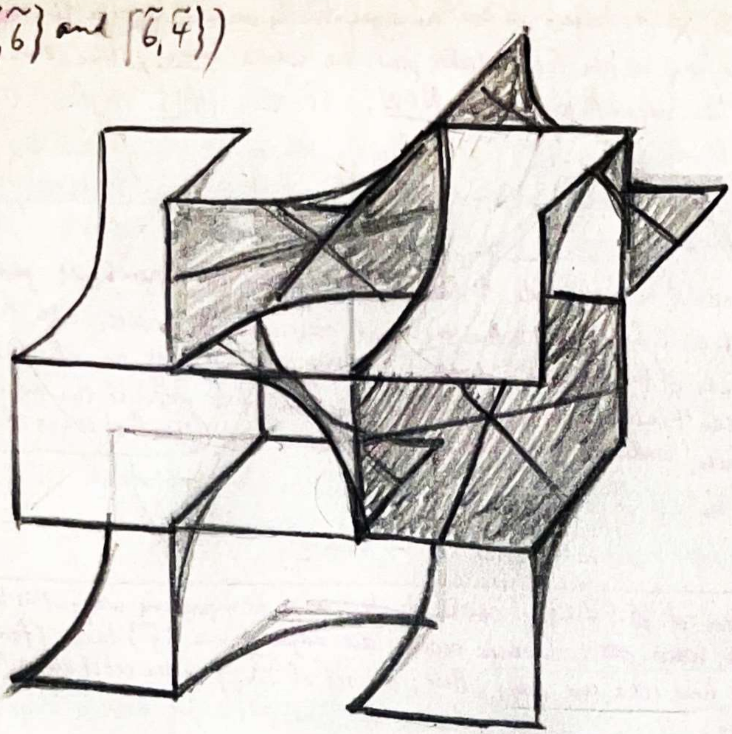
σB
σA
space-filling of ◊ tetrahedra η = −1/2
space-filling of ◊ tetrahedra η = 1/2
$\sigma_A = \sqrt{2}\left(\frac{1+\eta}{1-2\eta}\right)$
$\sigma_B = \sqrt{2}\left(\frac{1-\eta}{1+2\eta}\right)$
$\sigma\{\tilde{6}, 6\}_p$
-.25
-.2
-.1
η=0
.1
.2
.25
-.75

Space lattice is b.c.c.

(as with $\{\tilde{4},\tilde{6}\}$ and $\{\tilde{6},\tilde{4}\}$)

$\boxed{\{\tilde{6},\tilde{6}\}_L}$

dihedral angle $= \frac{\pi}{2}$



Fundamental region of translation group = tetrahedral assembly of 4 90° skew hexagons. (3 are shown shaded here)

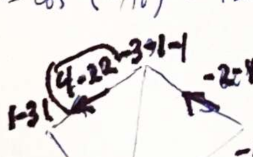
(These four share no common edges — only a single common vertex.)

$N\{\tilde{4},\tilde{6}\} = \{\tilde{6},\tilde{6}\}$

$M\{\tilde{6},\tilde{6}\} = \begin{Bmatrix}6\\ \tilde{6}\end{Bmatrix}$

~~Holes in $\{\tilde{p},\tilde{q}\}$~~

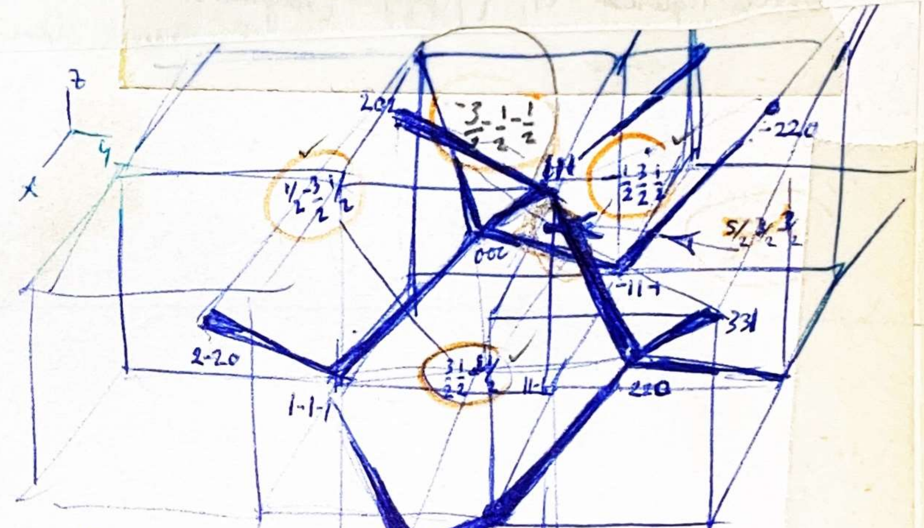
If there were a $\{\tilde{4},\tilde{6}\}$ reciprocal to the self-intersecting $\{\tilde{6},\tilde{4}\}$, it would have 112 edges with a face angle of 80.407° $= \cos^{-1}\left(\frac{1}{6}\right)$. $\sigma = \sqrt{1/5} = .4472$ Vertex figures would be $\{\tilde{6}\}$ with face angle $= \cos^{-1}(3/16) = 72.54°$

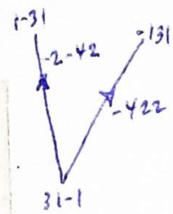


$\frac{-1\,2\,1 \quad 3\,1\,1}{-2+2+1}$ / 6 $= \frac{1}{6}$

1-30
301
300
10

$\varphi = 80.407°$





$-1-2\,1$
$-2\,1\,1$
$\frac{2-2+1}{6} = \frac{1}{6}$

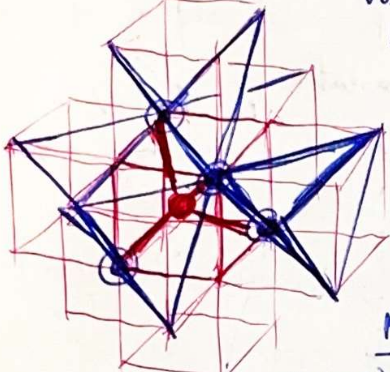
THE SELF-INTERSECTING $\{\tilde{6},\tilde{4}\}$

$\cos^{-1}\left(\frac{1}{6}\right) = 80.407°$

$\left[\cos^{-1}\left(-\frac{1}{6}\right) = 99.593°\right]$

36

12-24-72 notes added to discussion on p. 41 The pattern of collapse trajectories of the vertices of $\{6,6\}_P$
can be described as follows: Consider the vertices of the map $\{\tilde{6},6\}_P$ to be the set of vertices of a space-filling
of 10-hedra (6 $\{\tilde{4}\}$ and 4 $\{6\}$), the tetrahedrally symmetric saddle polyhedra whose centers describe a b.c.c. lattice
Each such 10-hedron has four vertices at ✱ the centers of the six symmetrically meeting edges. If these four vertices
are counted once each for all the 10-hedra, then all the f.c.c. lattice points are counted twice, since each such vertex
is shared by the ✱ star junctions of two adjacent 10-hedra. NOW: Consider HALF of the 10-hedra,
viz, those which share the $\{\tilde{6}\}$ faces. Assembled, they form the Schwarz D,
because their $\{\tilde{4}\}$ faces are left exposed. Inside each such 10-hedron, the 4 ✱
vertices converge to form a single vertex, as illustrated at the left.

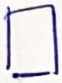


It would be worthwhile to compute the generalized Voronoi cell ~~just~~ for the
vertex configuration just before the final coalescence of vertices into the
vertices of the diamond network. It might have a large no. of faces.
I can't make any estimates from inspection of a model of the displaced
vertices, constructed from Pearce's Superstructure kit. (The final cell is of course the 16-hedron for D.)

Note: The set of $\{\vec{\Lambda}_i\}$ for this transformation is the set of all OUTWARD (or
INWARD) normal vectors at the vertices of of $\{\tilde{4},6\}_D$.

The pattern of collapse trajectories of the vertices of ~~4~~ $\{\tilde{4},6\}_D$ can be described in a corresponding way: this time, the
decahedra inside of which tetrahedrally symmetric vertex ~~...~~ coalescence occurs are adjacent via $\{\tilde{4}\}$ faces (forming a cellularized
labyrinth of P, with the $\{\tilde{6}\}$ faces exposed) (The final cells are cubes.) Here, the set of $\{\vec{\Lambda}_i\}$ is the set of all OUTWARD
(or INWARD) normal vectors at the vertices of $\{\tilde{6},6\}_P$. Again, just before the end, the final cells might have a large # of faces.

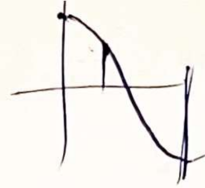
Vertex figures of $\{\tilde{p},q\}$ [regular saddle polyhedra
[puckered plane tessellations]

		Same as vertex figures of	
$\{\tilde{4},4\}$			$\{4,4\}$
$\{\tilde{6},3\}$			$\{6,3\}$
$\{\frac{\tilde{6}}{2},6\}$			$\{3,6\}$

$\{p,q\}$	$\{p,\tilde{q}\}$	$\{\tilde{p},q\}$	$\{\tilde{p},\tilde{q}\}$	$\left\{\begin{smallmatrix}p\\ q\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}p\\ \tilde{q}\end{smallmatrix}\right\}$ (or $\left\{\begin{smallmatrix}\tilde{p}\\ q\end{smallmatrix}\right\}$)	$\left\{\begin{smallmatrix}\tilde{p}\\ \tilde{q}\end{smallmatrix}\right\}$
$\boxed{3,3}$				$\boxed{\binom{3}{3}}$		
$\boxed{3,4}$				$\binom{3}{4}$	$\binom{3}{\tilde{4}}$ ✓	
$\boxed{3,5}$				$\binom{3}{5}$		
$3,6$		$\frac{\tilde{6}}{2},6$		$\binom{3}{6}$		
$\boxed{4,3}$				$\boxed{\binom{4}{3}}$		
$4,4$		$\tilde{4},4$		$\boxed{\binom{4}{4}}$		
	$4,\tilde{6}$	$\tilde{4},6$	$\tilde{4},\tilde{6}$	$\binom{4}{6}$ ✓	$\binom{4}{\tilde{6}}$ ✓	$\binom{\tilde{4}}{\tilde{6}}$ ✓
$\boxed{5,3}$						
$6,3$		$\tilde{6},3$		$\boxed{\binom{6}{3}}$		
	$6,\tilde{4}$	$\tilde{6},4$	$\tilde{6},\tilde{4}$	$\boxed{\binom{6}{4}}$	$\binom{6}{\tilde{4}}$ ✓	
	$6,\tilde{6}$	$\tilde{6},6$	$\tilde{6},\tilde{6}$	$\binom{6}{6} = \{6,\tilde{4}\}$	$\binom{6}{\tilde{6}}$	
$\left\{\frac{5}{2},5\right\}$						
$\left\{5,\frac{5}{2}\right\}$						
$\left\{\frac{5}{2},3\right\}$						
$\left\{3,\frac{5}{2}\right\}$						

38

$$\sigma = \tan\theta = \frac{\sin \varphi/2}{\sin \varphi_0/2} = \sqrt{\frac{1-\cos\varphi}{1-\cos\varphi_0}}$$



$\cos(99.953°) = -\frac{1}{6}$

$\cos(\varphi_0 = 120°) = -\frac{1}{2}$

$$\therefore \sigma_{99.553°\ \text{hex}} = \sqrt{\frac{1-(-1/6)}{1-(-1/2)}} = \sqrt{\frac{7/6}{3/2}} = \frac{7}{6}\,\frac{2}{3} = \sqrt{\frac{7}{9}}$$

$\cos(90°) = 0$

$\cos 120° = -\frac{1}{2}$

$$\therefore \sigma_{90°\ \text{hex}} = \sqrt{\frac{1}{3/2}} = \sqrt{2/3} = \frac{\sqrt{2}\sqrt{3}}{3} = \frac{\sqrt{6}}{3}$$

If edge is computed with "coordinates in smallest integers", then to obtain new expressions for $\cos\varphi$ & σ in terms of ζ, replace η by $\eta = r\zeta$ where

$r = \dfrac{\text{ratio of computed length of Old edge}}{\text{computed length of new edge}}$

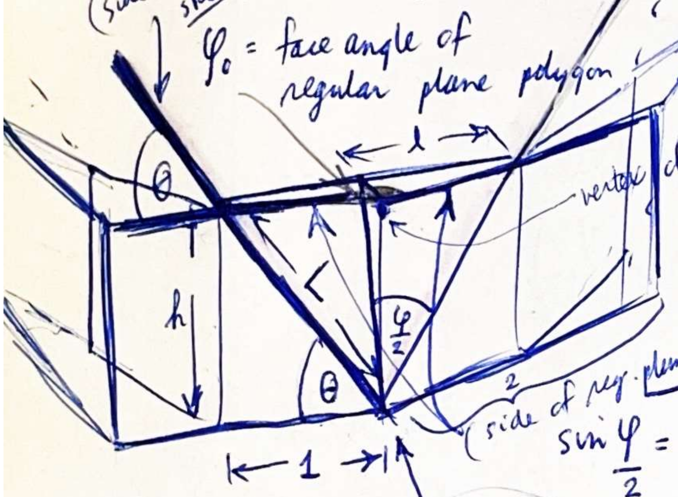
["new" means using coordinates in smallest integers")

Proof of relation $\boxed{\sigma = \text{skewness} = \tan\theta = \sqrt{\dfrac{\cos\varphi - \cos\varphi_0}{1-\cos\varphi}}}$

(side of reg. skew polygon)

φ_0 = face angle of regular plane polygon

φ = face angle of regular skew polygon

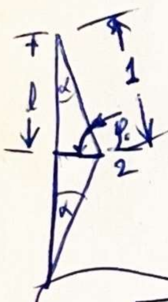


$L \sin\theta = h = \tan\theta$

$L\cos\theta = 1$

$\boxed{L = \dfrac{1}{\cos\theta}}$

$$\sin\frac{\varphi}{2} = \frac{l}{L} = \frac{\sin\frac{\varphi_0}{2}}{1/\cos\theta}$$



$\tan\theta = h$

$\alpha = \frac{\pi}{2} - \frac{\varphi_0}{2}$

$\cos\alpha = \sin\frac{\varphi_0}{2} = l$

$$\therefore \boxed{\cos\theta = \frac{\sin\frac{\varphi}{2}}{\sin\frac{\varphi_0}{2}}} = \sqrt{\frac{1-\cos\varphi}{1-\cos\varphi_0}}$$

$$\boxed{\sigma = \sqrt{\frac{\cos\varphi + \cos(2\pi/p)}{1-\cos\varphi}}}$$

$$\therefore \sigma = \tan\theta = \frac{\sin\theta}{\cos\theta} = \sqrt{\frac{\cos\varphi - \cos\varphi_0}{1-\cos\varphi}}$$

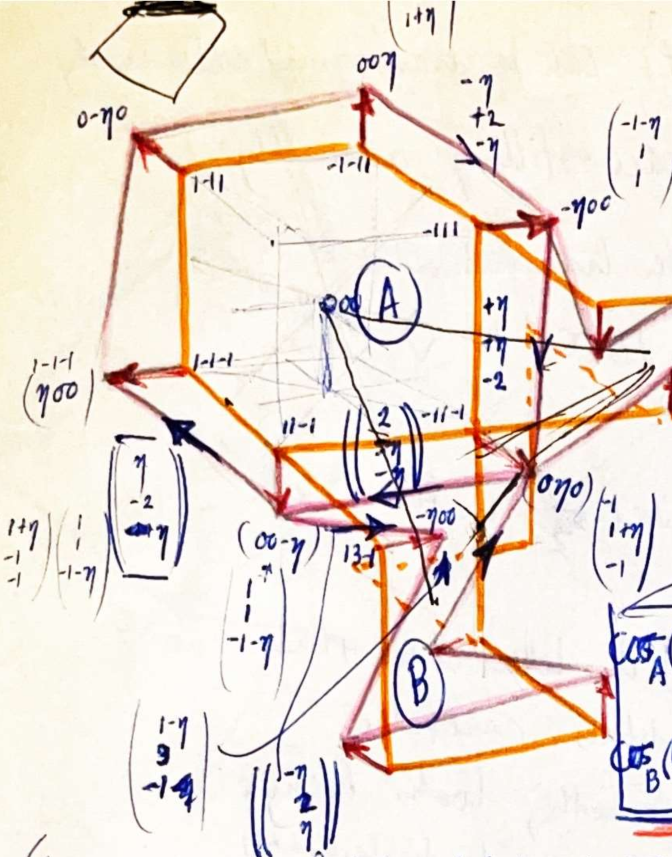
$$\boxed{\underset{\text{skewness}}{\sigma} = \tan\theta = \left(\frac{\cos\varphi - \cos\varphi_0}{1-\cos\varphi}\right)^{1/2}}$$

valid for any regular skew polygon, including skew star polygon

$$\boxed{\varphi = \cos^{-1}\left(\frac{\sigma^2 + \cos\varphi_0}{\sigma^2 + 1}\right)} \checkmark$$

$$\sin\theta = \left(\frac{1-\cancel{\cos\varphi_0}}{1-\cos\varphi}\right)^{1/2} \quad \cos\theta = \cancel{\left(\frac{-1+\cos\varphi_0}{1-\cos\varphi}\right)^{1/2}}$$

Transformation of $\{\tilde{6}[\tfrac{1}{2}]^{1/2}, 4\}$ $\longrightarrow$ into $\left\{\begin{matrix}\tilde{6}[\sigma_1] \\ \tilde{6}[\sigma_2]\end{matrix}\right\}_4$



If edge is in "smallest integers" then $\ell = 2$

$$\boxed{\begin{aligned}\cos_A(\varphi) &= \frac{\zeta^2-2\zeta}{2\zeta^2+1} = \frac{\zeta(\zeta-2)}{2\zeta^2+1} \\ \cos_B(\varphi) &= \frac{\zeta(\zeta+2)}{2\zeta^2+1}\end{aligned}} \quad ☆$$

~~Normalizing rule: If edge is of length ℓ, replace η by $\ell\zeta$~~ ☆

Start using this

~~If edge is normalized to ~~ ~~of small integers~~

$$\sigma_A(\zeta) = \frac{\sqrt{2}}{2}\left(\frac{2\zeta-1}{\zeta+1}\right) \checkmark$$

$$\sigma_B(\zeta) = \frac{\sqrt{2}}{2}\left(\frac{2\zeta+1}{\zeta-1}\right)$$

CORRECT ☆

(Last night, I verified by detailed calculations that $\left\{\begin{matrix}\tilde{6}[(\tfrac{1}{2})^{1/2}] \\ 4\end{matrix}\right\}$ transforms continuously into $\left\{\begin{matrix}6 \\ \tilde{4}[(2/5)^{1/2}]\end{matrix}\right\}$.)

In above figure, calculate

$$(\cos\varphi_6)_A = \frac{\begin{pmatrix}\eta\\-2\\\eta\end{pmatrix}\cdot\begin{pmatrix}-2\\\eta\\\eta\end{pmatrix}}{\eta^2+4+\eta^2} = \frac{-2\eta-2\eta+\eta^2}{2\eta^2+4} = \frac{\eta^2-4\eta}{2\eta^2+4} \quad A$$

$$\boxed{=\frac{\eta}{2}\left(\frac{\eta-4}{\eta^2+2}\right) = \cos\varphi_6}$$

$$\therefore \sigma_6(\eta) = \left(\frac{\left[\frac{\eta}{2}\left(\frac{\eta-4}{\eta^2+2}\right)+\frac{1}{2}\right]}{1-\frac{\eta}{2}\left(\frac{\eta-4}{\eta^2+2}\right)}\right)^{1/2} = \left(\frac{\frac{\eta^2-4\eta}{2\eta^2+4}+\frac{1}{2}}{1-\frac{\eta^2-4\eta}{2\eta^2+4}}\right)^{1/2} = \left(\frac{\frac{2\eta^2-8\eta+2\eta^2+4}{4\eta^2+8}}{\frac{2\eta^2+4-\eta^2+4\eta}{2\eta^2+4}}\right)^{1/2} = \left(\frac{\frac{4\eta^2-8\eta+4}{2(2\eta^2+4)}}{\frac{\eta^2+4\eta+4}{(2\eta^2+4)}}\right)^{1/2} = \left(\frac{2(\eta^2-2\eta+1)}{(\eta+2)^2}\right)^{1/2}$$

$$\text{or}\quad \sigma_6(\eta) = \left[\frac{2(\eta-1)^2}{(\eta+2)^2}\right]^{1/2} \qquad \boxed{\sigma_6^A(\eta) = \sqrt{2}\left[\frac{(\eta-1)}{(\eta+2)}\right]}$$

(vanishes for $\eta = 1$) ✓

$$\boxed{\sigma_6^A(\eta) = \sqrt{2}\left(\frac{\eta-1}{\eta+2}\right)} \checkmark$$

Now calculate $\cos\varphi_6 = \begin{pmatrix}-\eta\\2\\\eta\end{pmatrix}\cdot\begin{pmatrix}-2\\\eta\\\eta\end{pmatrix}\Big/ 2\eta^2+4 = \frac{2\eta+2\eta+\eta^2}{2\eta^2+4} = \frac{\eta^2+4\eta}{2\eta^2+4}$ B

for skewing hexagon, as check

$$\sigma_6(\eta) = \left(\frac{\cos\varphi_6 - \cos\varphi_c}{1-\cos\varphi}\right)^{1/2} = \left(\frac{\frac{\eta^2+4\eta}{2\eta^2+4}+\frac{1}{2}}{1-\frac{\eta^2+4\eta}{2\eta^2+4}}\right)^{1/2} = \left(\frac{\frac{2\eta^2+8\eta+2\eta^2+4}{4\eta^2+8}}{\frac{2\eta^2+4-\eta^2-4\eta}{2\eta^2+4}}\right)^{1/2} = \left(\frac{\frac{1}{2}(4\eta^2+8\eta+4)}{\eta^2-4\eta+4}\right)^{1/2} = \left(\frac{2(\eta^2+2\eta+1)}{(\eta-2)^2}\right)^{1/2}$$

~~$= \frac{\eta^2+4\eta}{2\eta^2+4} \Big/ \frac{2\eta^2+4-\eta^2-4\eta}{2\eta^2+4} = \frac{\eta^2+4\eta}{\eta^2-4\eta+4} = \frac{\eta(\eta+4)}{(\eta-2)^2}$ ~~ ~~$\sigma(\eta) = \ldots$~~

$$\boxed{\sigma_6^B(\eta) = \sqrt{2}\,\frac{(\eta+1)}{(\eta-2)}}$$

vanishes for $\eta = -1$ ✓

Checks ✓

When $\eta = \frac{1}{2}$, $\sigma_B = \sqrt{2}$; $\sigma_A = -\frac{\sqrt{2}}{5}$: $\varphi_A = \cos^{-1}(-7/18) = 112.885$

Clearly $\sigma_6^A(\eta) = \sigma_6^B(\eta)$ only when $\eta = 0$.

When $\eta = 1$, $\sigma_6^A = 0$; $\sigma_6^B = \sqrt{8}$ ($\varphi = 33.559°$) Thus, $\{\tilde{6}, 4\} \longrightarrow \left\{\begin{matrix}\tilde{6}\\6\end{matrix}\right\}$

$$\boxed{\sigma_6^B(\eta) = \sqrt{2}\left(\frac{\eta+1}{\eta-2}\right)}$$

46

The diamond IPMS $\left\{\tilde{6}(1/2)^{1/2}, 4\right\}$ can be transformed continuously either into $\left\{\begin{matrix} 6 \\ \tilde{6}(8)^{1/2} \end{matrix}\right\}$ or into a space-filling assembly of tetragonal tetrahedra! (This latter transformation can be described as a change from IBCC to ◇ net, equally well.)

These transformations are ~~well~~ conveniently described ~~by~~ in terms of σ (skewness).

$$\sigma_6^A(\eta) = \sqrt{2}\left(\frac{\eta-1}{\eta+2}\right)$$

$$\sigma_6^B(\eta) = \sqrt{2}\left(\frac{\eta+1}{\eta-2}\right)$$

When $\eta = 2$, B-hexagons disappear, and A-hexagons have $\sigma_6^A(2) = \sqrt{1/8}$ (i.e., we have a space filling of diamond ~~tetragonal~~ tetrahedra).

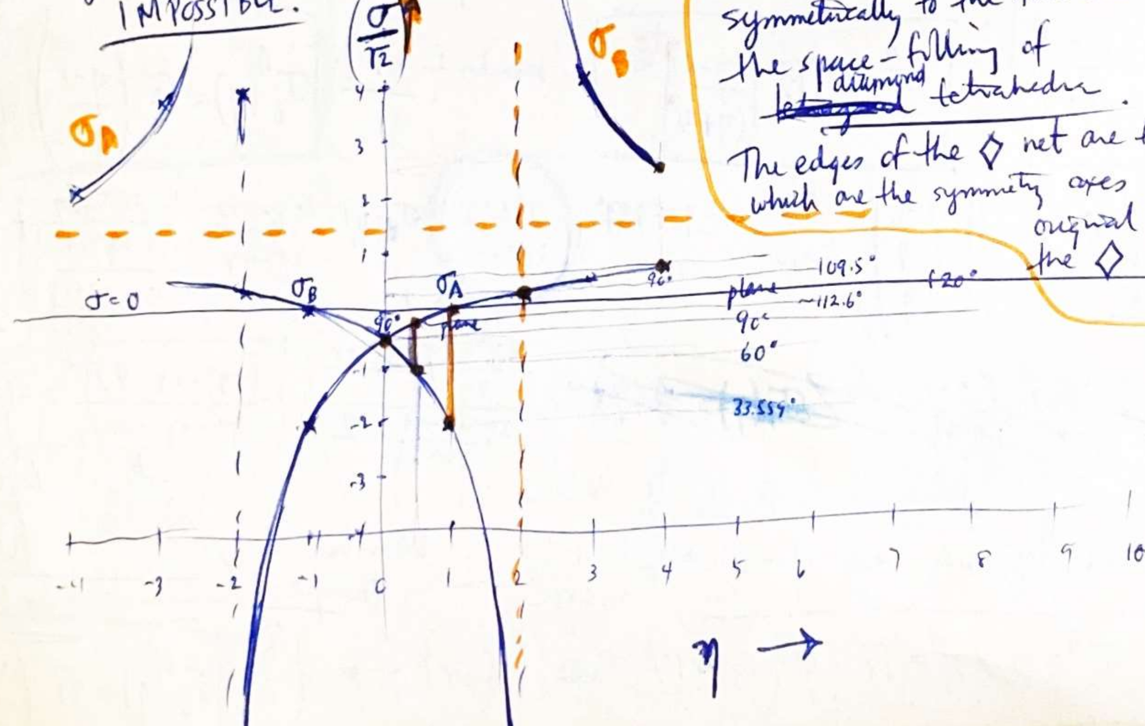
Thus, one diamond-labyrinth has grown, while the other has disappeared?: IMPOSSIBLE!

I'd better check these details carefully. Something looks fishy. After all, the displacements η take place equally often into each of the 2 labyrinths. Nevertheless, it is true that as η goes from 0 to 4

NOTHING FISHY!

The 2 labyrinths each contribute symmetrically to the formation of the space-filling of ~~tetragonal~~ diamond tetrahedra.

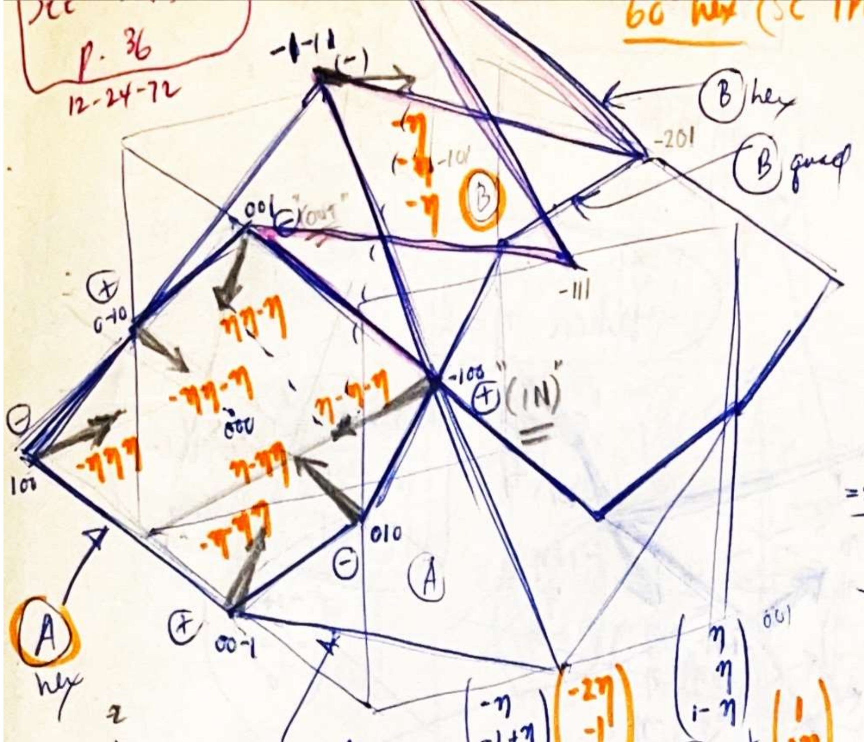
The edges of the ◇ net are the lines which are the symmetry axes of $\frac{1}{2}$ the original faces of the ◇ IPMS!

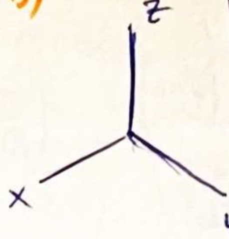


see p. 36
12-24-72

60° hex (SC TPMS)

Variable skewness $\left\{\begin{matrix}\tilde{6}\\ \tilde{6}\end{matrix}\right\}_6$





$$\cos\varphi_6^A = \begin{pmatrix}1\\2\eta\\1\end{pmatrix}\cdot\begin{pmatrix}2\eta\\1\\1\end{pmatrix}\Big/ 2+4\eta^2$$

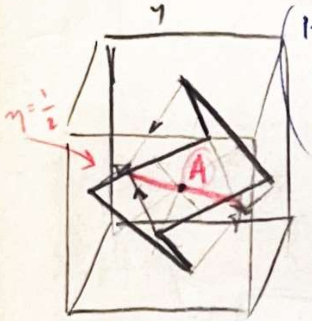
$$=\frac{2\eta+2\eta+1}{2+4\eta^2}=\frac{1+4\eta}{2+4\eta^2}=\frac{4\eta+1}{4\eta^2+2}=\boxed{\frac{1+4\eta}{2(1+2\eta^2)}}$$

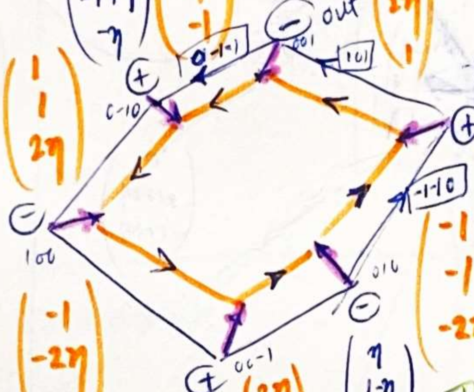
✓ $=\frac{1}{2}$ or ($\theta=60°$) if $\eta\to 0$

$\cos\varphi_6^B$

-1-11
$-\eta-\eta-\eta$
$\begin{pmatrix}-1-\eta\\-1-\eta\\1-\eta\end{pmatrix}$
A
$\begin{pmatrix}-2\eta\\-1\\1\end{pmatrix}$
001
$\eta\eta-\eta$
$\boxed{\eta\eta 1-\eta}$
$\begin{pmatrix}-1\\-2\eta\\-1\end{pmatrix}$
-100
$\eta-\eta-\eta$
$\begin{pmatrix}-1+\eta\\-\eta\\-\eta\end{pmatrix}$
$\begin{pmatrix}-1+\eta\\-\eta\\-\eta\end{pmatrix}$

$$\boxed{\begin{aligned}\sigma_A&=\sqrt{2}\left(\frac{1+\eta}{1-2\eta}\right)\\ \sigma_B&=\sqrt{2}\left(\frac{1-\eta}{1+2\eta}\right)\end{aligned}}$$





$\begin{pmatrix}\eta\\\eta\\1-\eta\end{pmatrix}$ $\begin{pmatrix}-\eta\\-1+\eta\\-\eta\end{pmatrix}$ $\begin{pmatrix}-2\eta\\-1\\-1\end{pmatrix}$ $\begin{pmatrix}1\\2\eta\\1\end{pmatrix}$ $\begin{pmatrix}1\\1\\2\eta\end{pmatrix}$ $\begin{pmatrix}1-\eta\\\eta\\\eta\end{pmatrix}$ $\begin{pmatrix}-1+\eta\\-\eta\\-\eta\end{pmatrix}$ $\begin{pmatrix}-1\\-1\\-2\eta\end{pmatrix}$ $\begin{pmatrix}-1\\-2\eta\\-1\end{pmatrix}$ $\begin{pmatrix}\eta\\1-\eta\\\eta\end{pmatrix}$ $\begin{pmatrix}2\eta\\1\\1\end{pmatrix}$ $\begin{pmatrix}-\eta\\-\eta\\-1+\eta\end{pmatrix}$

This leads to a symmetric binary decomposition of the f.c.c. lattice

$$\sigma_A^{(6)}=\sqrt{\frac{\cos\psi-\cos\varphi_6}{1-\cos\psi}}=\left[\frac{\frac{1+4\eta}{2+4\eta^2}+\frac{1}{2}}{1-\frac{1+4\eta}{2+4\eta^2}}\right]^{1/2}=\left[\frac{\frac{2+8\eta+2+4\eta^2}{2(2+4\eta^2)}}{\frac{2+4\eta^2-1-4\eta}{2+4\eta^2}}\right]^{1/2}=\left[\frac{1}{2}\,\frac{4\eta^2+8\eta+4}{4\eta^2-4\eta+1}\right]^{1/2}=\sqrt{2}\left[\frac{\eta^2+2\eta+1}{(2\eta-1)^2}\right]^{1/2}=\boxed{\sqrt{2}\left(\frac{\eta+1}{2\eta-1}\right)}$$

$\sigma_A^{(6)}$

$$\cos\varphi_B = \frac{\begin{pmatrix}-1\\-2\eta\\-1\end{pmatrix}\cdot\begin{pmatrix}2\eta\\1\\-1\end{pmatrix}}{2+4\eta^2}=\frac{-2\eta-2\eta+1}{2+4\eta^2}=\frac{1-4\eta}{2+4\eta^2}$$

$$\sigma_B^{(6)}=\sqrt{\frac{\frac{1-4\eta}{2+4\eta^2}+\frac{1}{2}}{1-\frac{1-4\eta}{2+4\eta^2}}}=\sqrt{\frac{\frac{2-8\eta+2+4\eta^2}{2(2+4\eta^2)}}{\frac{2+4\eta^2-1+4\eta}{2+4\eta^2}}}=\sqrt{\frac{4\eta^2-8\eta+4}{2(4\eta^2+4\eta+1)}}$$

$\sigma_B^{(6)}$

$$=\sqrt{2}\sqrt{\frac{\eta^2-2\eta+1}{2\eta+1}}=\boxed{\sqrt{2}\left(\frac{\eta-1}{2\eta+1}\right)}$$

If $\eta=\frac{1}{2}$: $\sigma_A\to\infty$, $\sigma_B\to 109.5°$ (diamond tetrahedron)

If $\eta=1$, $\sigma_A=\sqrt{8}$ (33.53° hex), $\sigma_B=0$

$$\cancel{\cos\varphi_A^{(4)}=\cos_A^{(6)}=\frac{1+4\eta}{2+4\eta^2}\quad \cos\varphi_B^{(4)}=\cos\varphi_B^{(6)}=\frac{1-4\eta}{2+4\eta^2}}$$

$$\cancel{\sigma_A^{(4)}=\left[\frac{\frac{1+4\eta}{2+4\eta^2}}{\frac{(2\eta-1)^2}{2+4\eta^2}}\right]^{1/2}=\left(\frac{1}{2\eta-1}\right)\left[\frac{4\eta+1}{4\eta^2+2}\right]^{1/2}=\frac{(4\eta+1)^{1/2}}{2\eta-1}\quad \sigma_B^{(4)}=\left[\frac{\frac{1-4\eta}{2+4\eta^2}}{\frac{(2\eta+1)^2}{2+4\eta^2}}\right]^{1/2}\quad \sigma_B^{(4)}=\frac{(1-4\eta)^{1/2}}{2\eta+1}}$$

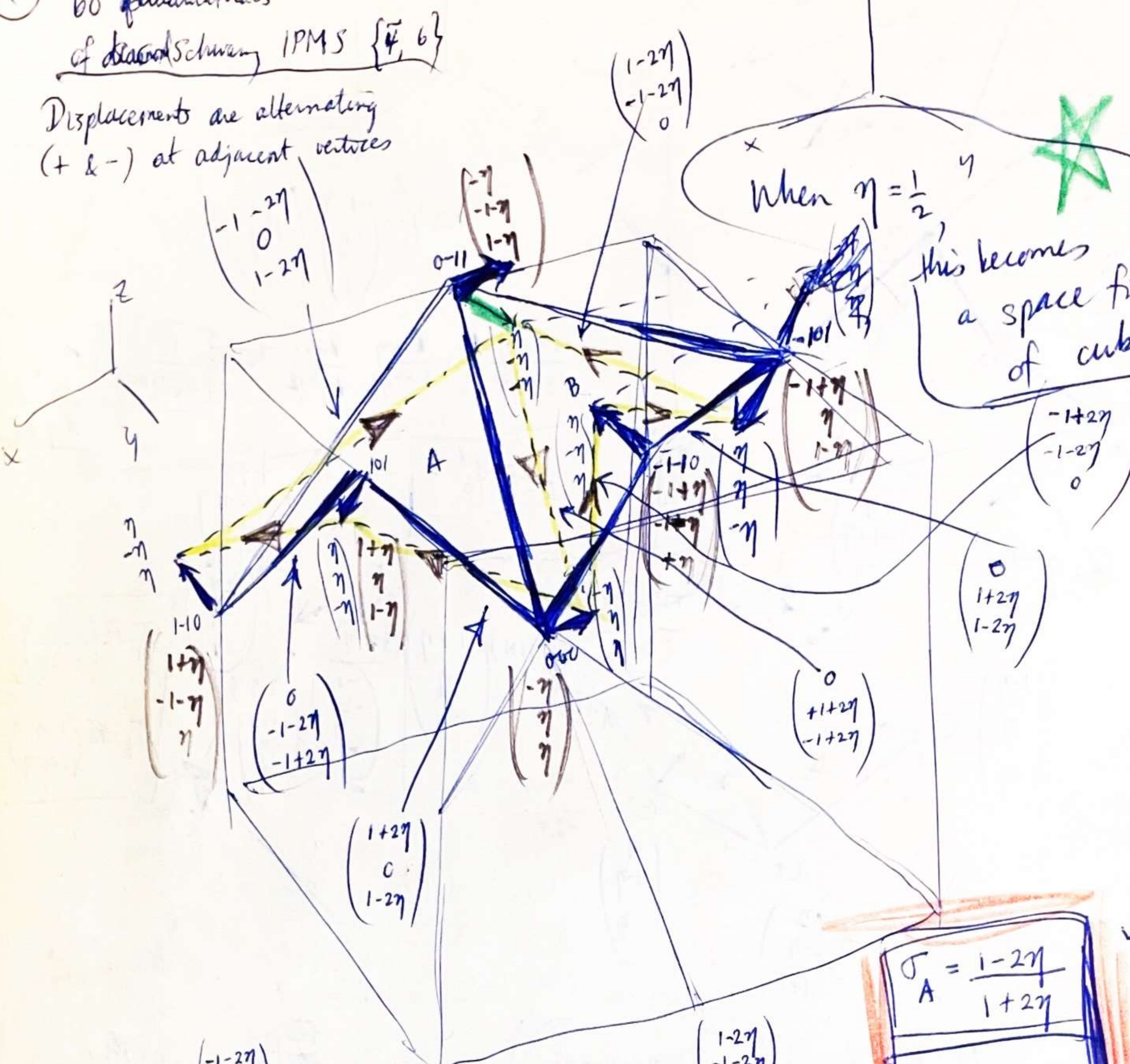
~~But this does not~~ preserve regularity of both A & B quadrilaterals

(42)

60° quadrilaterals of ~~diamond~~ Schwarz IPMS $\{\bar{4}, 6\}$

Displacements are alternating (+ & −) at adjacent vertices

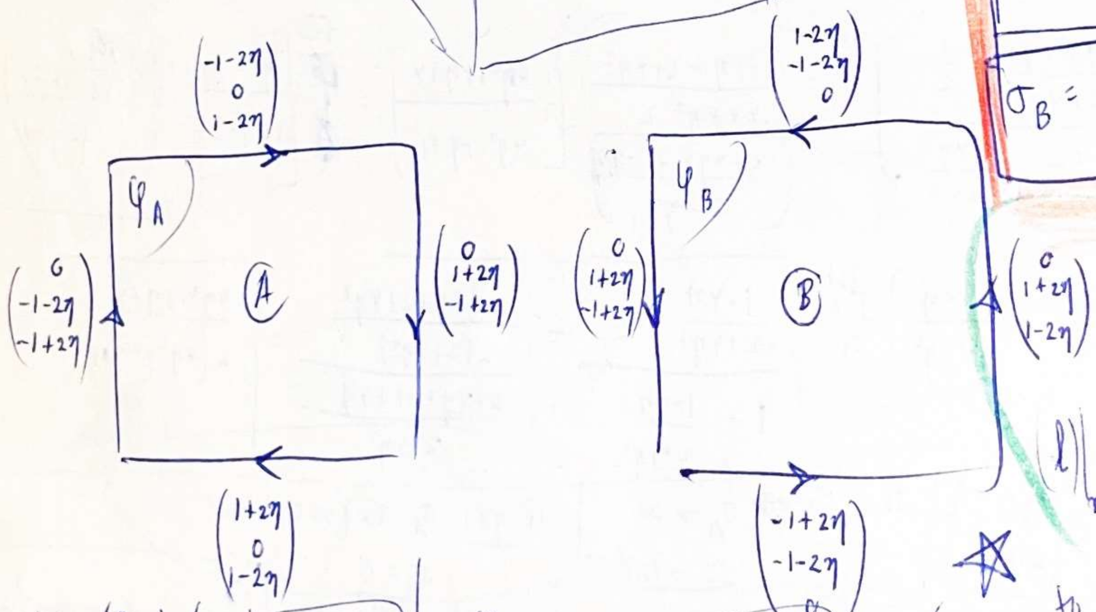
Variable ... $\{4\}_6$



When $\eta = \frac{1}{2}$, this becomes a space filling of cubes

$$\boxed{\sigma_A = \frac{1-2\eta}{1+2\eta}}$$

$$\boxed{\sigma_B = \frac{1+2\eta}{1-2\eta}}$$



When $\eta = 1/2$,

$$\boxed{\begin{aligned}\sigma_A &= 0\\ \sigma_B &\to \infty\end{aligned}}$$

$$l\Big|_{\eta=1/2} = \sqrt{2+8\eta^2}\Big|_{\eta=1/2} = \sqrt{2+8\cdot\tfrac{1}{4}} = 2$$

Cf. original length $= \sqrt{2}$

$\therefore$ length has increased in ratio $\sqrt{2}$.

$$\cos\varphi_4^{(A)} = \frac{\begin{pmatrix}0\\-1-2\eta\\1-2\eta\end{pmatrix}\cdot\begin{pmatrix}1+2\eta\\0\\1-2\eta\end{pmatrix}}{(1+2\eta)^2+(1-2\eta)^2} = \boxed{\frac{(1-2\eta)^2}{2+8\eta^2}}$$

$$\cos\varphi_4^{(B)} = \frac{\begin{pmatrix}1-2\eta\\1+2\eta\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1+2\eta\\-1+2\eta\end{pmatrix}}{(1+2\eta)^2+(1-2\eta)^2} = \boxed{\frac{(1+2\eta)^2}{2+8\eta^2}}$$

$\eta = 0$

$-\frac{1}{2}$ (60° quad)

(Turn to bottom of previous page)

Summary of transformations on 6-connected f.c.c. net.

$\frac{.283}{5\overline{)1.414}}$ (43)

60° hexagons

SC IPMS $\boxed{\{\tilde{6}[\sqrt{2}], 6\}}$

$$\sigma_A^{(6)} = \sqrt{2}\left(\frac{\eta+1}{2\eta-1}\right)$$

$$\sigma_B^{(6)} = \sqrt{2}\left(\frac{\eta-1}{2\eta+1}\right)$$

packing of ~~[illegible]~~

$\boxed{\left\{\begin{matrix}6\\ \tilde{6}[\sqrt{8}]\end{matrix}\right\}}$

$\left\{\begin{matrix}\tilde{6}[\sigma_A]\\ \tilde{6}[\sigma_B]\end{matrix}\right\}$

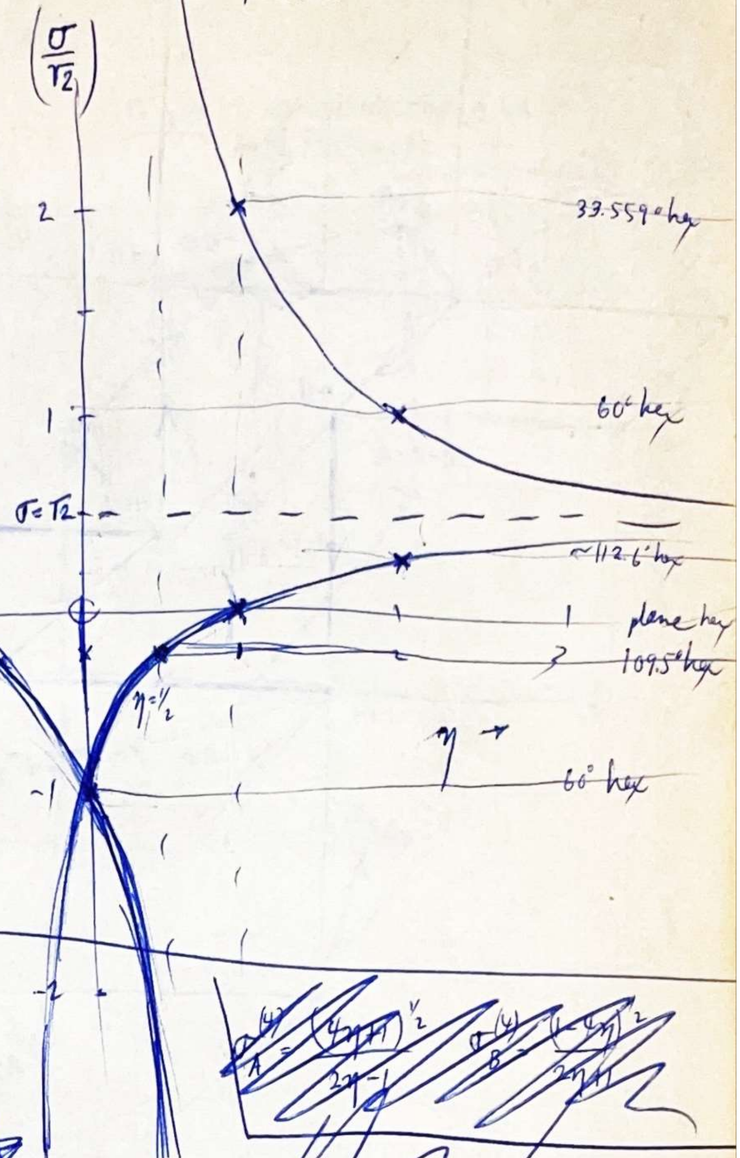
If $\eta = \frac{1}{2}$, $\sigma_A \to \infty$
$\sigma_B \to \sqrt{\frac{1}{8}}$ (109.5° hex)

If $\eta = 1$, $\sigma_A \to \sqrt{8}$ (33.559° hex)
$\sigma_B = 0$

If $\eta = 2$, $\sigma_A \to \sqrt{2}$ (60° hex.)
$\sigma_B \to \frac{\sqrt{2}}{5}$ (~112.6° hex

$\{\tilde{6}[\sqrt{2}], 4\}$

If $\eta = 0$, $\sigma_A \to -\sqrt{2}$ 60° hex
$\sigma_B \to -\sqrt{2}$



Transformations on 60° quadrilaterals

◇ IPMS $\{\tilde{4}[1], 6\}$

~~To carry these out in such a way as to preserve the regularity of the~~ ~~two~~ ~~quadrilaterals,~~ ~~all vertices~~ ☆
~~must be displaced inward [outward] from one labyrinth (S.C.) into the other.~~

NONSENSE!
(algebraic error)

~~$\sigma_A^{(4)} = \left(\frac{4\eta+1}{2\eta-1}\right)^{\frac{1}{2}}$~~ ~~$\sigma_B^{(4)} = \left(\frac{1-4\eta}{2\eta+1}\right)^{\frac{1}{2}}$~~

~~$\eta = \frac{1}{2}$ $\sigma_A \to \infty$~~
~~$\sigma_B \to$~~

$$\sigma_4^{(A)} = \sqrt{\frac{\cos\varphi - \cos\varphi_0}{1-\cos\varphi}} = \left[\frac{\frac{(1-2\eta)^2}{2+8\eta^2}}{1-\frac{(1-2\eta)^2}{2+8\eta^2}}\right]^{\frac{1}{2}} = \left[\frac{(1-2\eta)^2}{2+8\eta^2-1+4\eta-4\eta^2}\right]^{\frac{1}{2}} = \boxed{\frac{1-2\eta}{1+2\eta}}$$

$1-4\eta+4\eta^2$; $= 4\eta^2+4\eta+1$

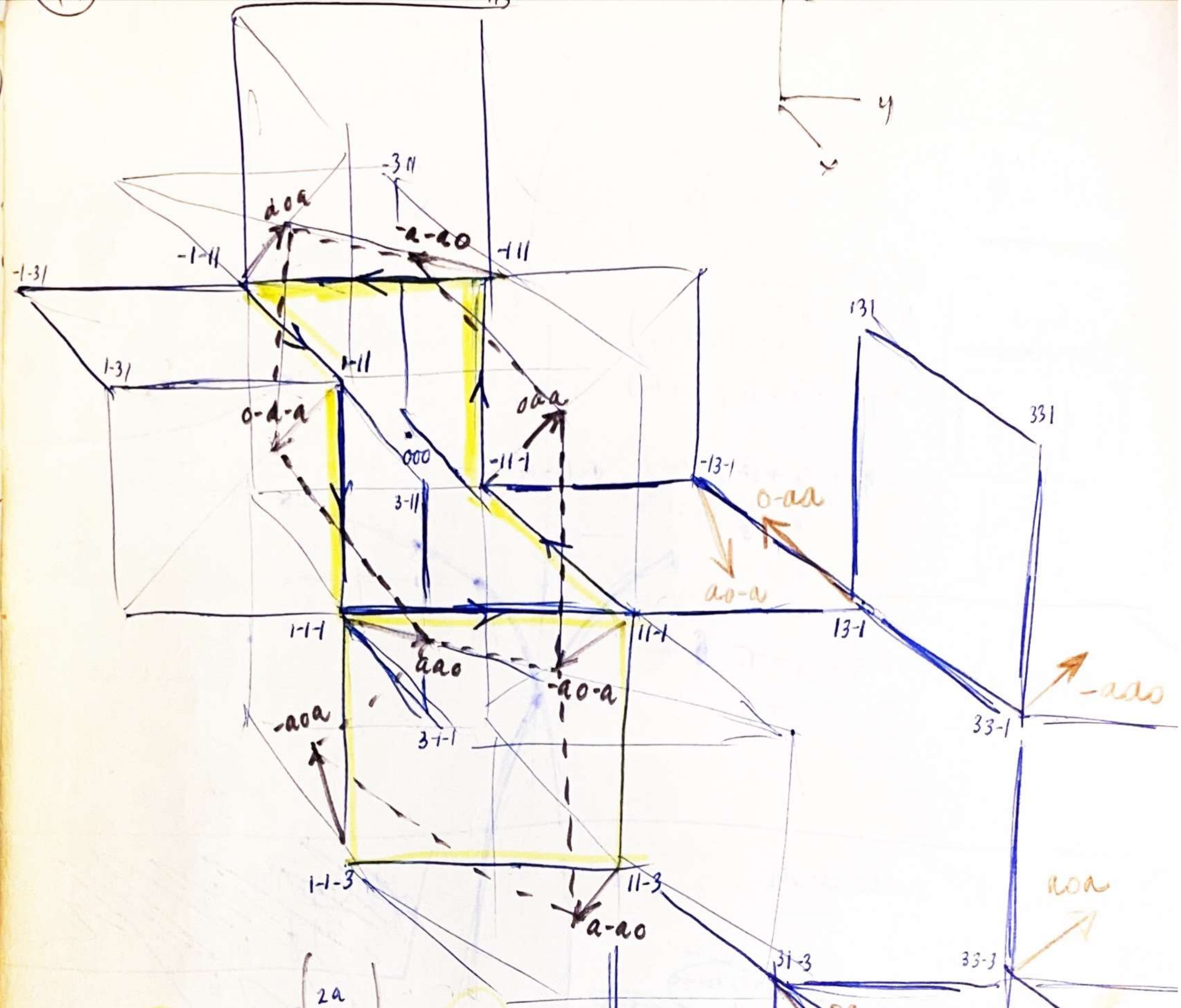
$$\sigma_4^{(B)} = \left[\frac{\frac{(1+2\eta)^2}{2+8\eta^2}}{1-\frac{[1+4\eta+4\eta^2]}{2+8\eta^2}}\right]^{\frac{1}{2}} = \frac{1+2\eta}{[2+8\eta^2-1-4\eta-4\eta^2]^{\frac{1}{2}}} = \boxed{\frac{1+2\eta}{1-2\eta}}$$

η	$\sigma_4^{(A)}$	$\sigma_4^{(B)}$
0	1 (60° quad)	1 (60° quad)
$\frac{1}{2}$	0 square	∞ line
$\frac{1}{4}$	$\frac{1}{3}$ ~86° quad	−3 ~32° quad

Presumably $\eta = \frac{1}{2}$ corresponds to $\{4, \tilde{6}[\sqrt{8}]\}_{S.C.}$

(☆ Check this point ↗)

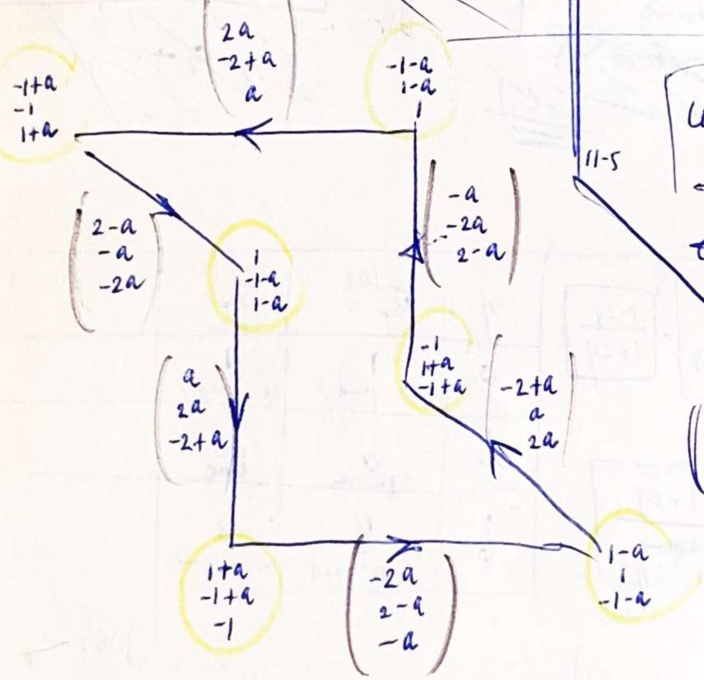
No, that's wrong.
It becomes a space-filling of cubes!

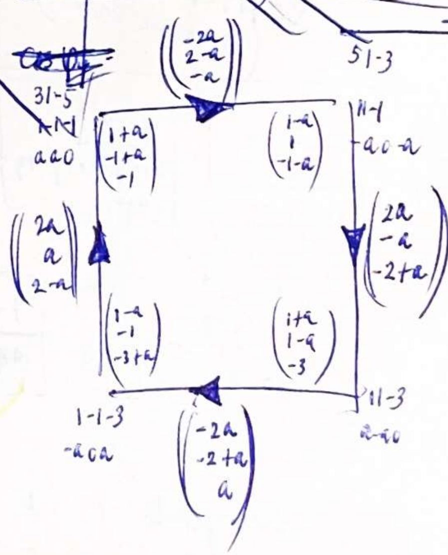


$$\cos\varphi_6 = \frac{5a^2-6a}{6a^2-4a+4}$$

$$\cos\varphi_4 = \frac{4a^2}{6a^2-4a+4}$$

~~$\cos\varphi$~~





-1/2 (60°)

...ral cases

$\cos^{-1}\left(-\frac{13}{30} = \cancel{.8666}\ -.43333\right) = 115.680°)$

Cf. p. 15 To normalize edge length of L.C. $(\hat{4}\cdot\hat{6})_L$ to $\{\bar{6},\bar{4}\}_L$, use: $\frac{\frac{\sqrt{14}}{2}}{\frac{\sqrt{14}}{4}} = \frac{x}{\sqrt{\frac{5}{6}}}$ $\therefore x = \frac{\sqrt{30}}{3} = \boxed{1.826}$

edge length $= \sqrt{6}$

$\varphi_6(\sigma)\Big|_{\sigma=\sqrt{\frac{2}{43}}}$ $\cancel{\cos^{-1}\left(\frac{27}{135} = .2\ldots\right) = 101.5\ldots°}$

$\left(l_{\eta=\frac{1}{6}} = \sqrt{1-2\eta+6\eta^2}\Big|_{\eta=\frac{1}{6}} = \frac{\sqrt{14}}{4}\right)$

$l_{\eta=\frac{1}{4}} = \sqrt{\frac{5}{6}})$

$\boxed{\text{Locally centered [12-5] configuration}}$

$\varphi_4(\sigma)\Big|_{\sigma=\sqrt{\frac{2}{13}}} = \cos^{-1}\left(+\frac{2}{15} = .1333\ldots\right) = 82.338°$

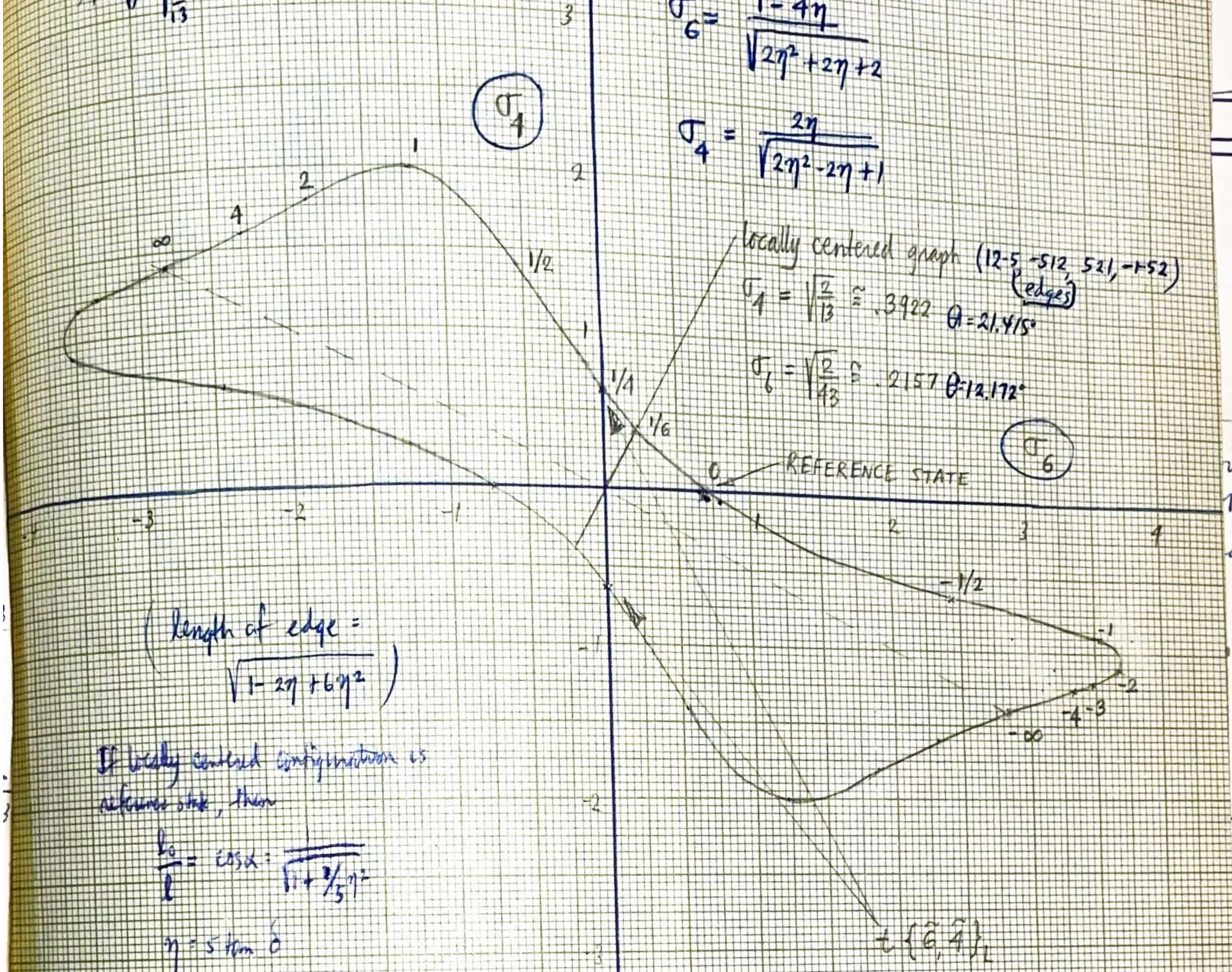
$$\sigma_6 = \frac{1-4\eta}{\sqrt{2\eta^2+2\eta+2}}$$

$$\sigma_4 = \frac{2\eta}{\sqrt{2\eta^2-2\eta+1}}$$

locally centered graph (12-5, -512, 521, -152) (edges)

$\sigma_4 = \sqrt{\frac{2}{13}} \cong .3922$ $\theta = 21.415°$

$\sigma_6 = \sqrt{\frac{2}{43}} \cong .2157$ $\theta = 12.172°$



(length of edge $= \sqrt{1-2\eta+6\eta^2}$)

If locally centered configuration is reference state, then

$\frac{l_0}{l} = \cos\alpha = \frac{1}{\sqrt{1+\frac{12}{5}\eta^2}}$

$\eta = 5\tan\delta$

ing
pplied to
ds

for $\eta = \frac{1}{2}$

$\sqrt{2\eta^2+2\eta+2}$ ✓ $\sigma_4 = \frac{2\eta}{\sqrt{2\eta^2-2\eta+1}}$ ✓

$\cos\varphi_4 = \frac{4\eta^2}{1-2\eta+6\eta^2}$

7-7-68

$\sigma_4 = 2\eta \Big/ \sqrt{2\eta^2 - 2\eta + 1}$

$\sigma_6 = \cancel{(4\eta-1)}\,(1-4\eta) \Big/ \sqrt{2\eta^2 + 2\eta + 2}$

η	σ_4	σ_6	$\sqrt{6}$
$-\infty$	$-\sqrt{2} = -1.414$	$+2\sqrt{2} = +2.828$	$2\sqrt{2} = 2.828$
-4	$-\frac{8\sqrt{41}}{41} = -1.25$	$+\frac{17\sqrt{30}}{30} = +3.10$	$\frac{17\sqrt{26}}{26} = 3.336$
-3	$-\frac{6}{5} = -1.2000$	$+13\sqrt{17}/17 = +3.15$; $\cancel{+\frac{17\sqrt{30}}{30} = +3.10}$	$\frac{13\sqrt{14}}{14} = 3.475$
-2	$-\frac{4\sqrt{13}}{13} = -1.10$	$+\frac{9\sqrt{2}}{4} = +3.183$	$\frac{3}{2}\sqrt{6} = 3.678$
-1	$-\frac{2\sqrt{5}}{5} = -.89$	$+\frac{5\sqrt{3}}{3} = +2.88$	$\frac{5\sqrt{2}}{2} = 3.535$
$-1/2$	$-.633 = -\sqrt{\frac{2}{5}}$	$+\frac{3\sqrt{2}}{2} = +2.12$	$\sqrt{6} = 2.449$
0	0	$+\frac{\sqrt{2}}{2} = +.707$	$+\frac{\sqrt{2}}{2} = .707$
$1/4$	$\frac{\sqrt{10}}{5} = \sqrt{\frac{2}{5}}$ .63	0	0
$1/2$	$\sqrt{2} = 1.414$	$-\frac{\sqrt{3}}{3} = -.577$	$-\sqrt{\frac{2}{7}} = -.534$
1	2	$-\frac{3\sqrt{5}}{5} = -1.341$	$-\frac{\sqrt{6}}{2} = -1.224$
2	$\frac{4\sqrt{5}}{5} = 1.79$	$-\frac{7\sqrt{3}}{6} = -2.02$	$-\frac{\sqrt{14}}{2} = -1.871$
3			
4	$\frac{8}{5} = 1.6000$	$-\frac{15\sqrt{38}}{38} = -2.435$	$-\frac{15\sqrt{42}}{42} = -2.313$
∞	$\sqrt{2} = 1.414$	$-2\sqrt{2} = -2.828$	$-2\sqrt{2} = -2.828$
$1/6$	$\sqrt{\frac{2}{13}} = .3922$		$\sqrt{\frac{2}{43}} = .2157$

2a

$\sqrt{\frac{2}{5}}$

re the

ted.

expressed

instead o

$\frac{\eta^2 - 3\cdot\eta}{2\eta + 6\eta^2}$

$\frac{4\eta^2}{1 - 2\eta + 6\eta^2}$

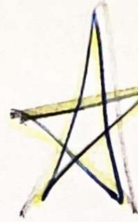
$2\,\cancel{(1+4\eta)}$

$\boxed{\sigma_6 = \frac{4\eta - 1}{\sqrt{2\eta^2 + \cancel{2\eta} + 2}}}$

7-7-68

We find that $\left\{ \begin{matrix} 4 \\ \tilde{6}\,[\sqrt{1/2}] \end{matrix} \right\}$ and $\left\{ \begin{matrix} 6 \\ \tilde{4}\,[\sqrt{2/5}] \end{matrix} \right\}$ are special cases

of $\left\{ \begin{matrix} \tilde{4}\,[\sigma_4] \\ \tilde{6}\,[\sigma_6] \end{matrix} \right\}$



$$\boxed{\sigma_6 = \frac{\sqrt{2}\,(2a-1)}{\sqrt{a^2+2a+4}} \qquad \sigma_4 = \frac{\sqrt{2}\,a}{\sqrt{a^2-2a+2}}}$$

When $a = 1/2$, $\begin{pmatrix} a \\ 2a \\ -2+a \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ -3/2 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

Set $\dfrac{5a^2-6a}{6a^2-4a+4} = \cos\varphi_6 = -\dfrac{1}{2} \Longrightarrow \boxed{a = \dfrac{1}{2}}$

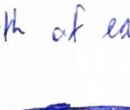
(i.e., assume that the skewness of the 90° hexagon vanishes ~~decreases~~, leading to the plane hexagon)

When $a = -2$, $\sigma_6 = -\frac{5\sqrt{2}}{2}$

$$\sigma_6 = \sqrt{\frac{\cos\varphi - \cos\varphi_0}{1-\cos\varphi}} = \sqrt{\frac{2(2a-1)^2}{\cancel{(2+a)^2}\; a^2+2a+4}} = \boxed{\frac{\sqrt{2}\,(2a-1)}{\cancel{(2+a)}\;\cancel{\sqrt{a^2+2a+4}}}}\Bigg|_{a=1/2} = 0 \; ✓$$

$$\sigma_4 = \cancel{\sqrt{\frac{2a^2-2a+2}{2a^2-4a+4}}} \; \boxed{\frac{\sqrt{2}\,a}{\sqrt{a^2-2a+2}}}\Bigg|_{a=1/2} = \sqrt{\frac{2}{5}} \; ✓$$

Thus, the skewing transformation applied to $\left\{ \begin{matrix} 4 \\ \tilde{6}\,[\sqrt{1/2}] \end{matrix} \right\}$ leads to $\left\{ \begin{matrix} 6 \\ \tilde{4}\,[\sqrt{2/5}] \end{matrix} \right\}$ for $\eta = \frac{1}{2}$

Kagome and  are related in the plane the way the cuboctahedron and N. Johnson's collapsing icosahedron are related.

If this figure is redrawn with the vertices expressed in terms of smallest integers, then the length of each edge becomes 1 instead of 2,

& ~~$\sigma_6 = \sqrt{2}\,(1+\eta-1)$~~ ~~$\sigma_6 = \dfrac{\sqrt{2}\,(-1+4\eta)}{2\,(1+\eta)}$~~ ; ~~$\sigma_4 = \dfrac{2\sqrt{2}\,\eta}{\sqrt{4\eta^2-4\eta+2}}$~~

$\cos\varphi_6 = \dfrac{5\eta^2-3\eta}{1-2\eta+6\eta^2}$

$\cos\varphi_4 = \dfrac{4\eta^2}{1-2\eta+6\eta^2}$

$$\boxed{\sigma_6 = \frac{4\eta-1}{\sqrt{2\eta^2+2\eta+2}}} \; ✓ \qquad \boxed{\sigma_4 = \frac{2\eta}{\sqrt{2\eta^2-2\eta+1}}} \; ✓$$

7-7-68

46

	Regular figure $\{\tilde{p},\tilde{q}\}$	Result of skewing transformation on $\{\tilde{p},\tilde{q}\}$	$t\{\tilde{p},\tilde{q}\}$	$h\{\tilde{p},\tilde{q}\}$	Skewing transformation on $t\{\tilde{p},\tilde{q}\}$	Hole (reg. polyg.)
A. D	$\eta=\frac{1}{2}\tan\alpha$ RST $\left\{\tilde{4}(\sqrt{1}),6\right\}_D$ 60° $\cos\alpha=(1+4\eta^2)^{-1/2}=\frac{l_c}{l}$	1 $\left\{\begin{matrix}\tilde{4}\left[\frac{1-2\eta}{1+2\eta}\right]\\ \tilde{4}\left[\frac{1+2\eta}{1-2\eta}\right]\end{matrix}\right\}$ 42, 3	2 $\left\{\begin{matrix}4\\6\end{matrix}\right\}_D$	$\{6,\tilde{6}[\sqrt{8}]\}_D$ 33.6°	$\left\{\begin{matrix}\tilde{4}\left[\sqrt{2}\eta/\sqrt{1+\eta^2}\right]\\ \tilde{6}\left[2\sqrt{2}\eta/\sqrt{3+\eta^2}\right]\end{matrix}\right\}$ 58; $\eta=\frac{\sqrt{3}}{3}\tan\alpha$ $\cos\alpha=(1+3\eta^2)^{-1/2}=\frac{l_c}{l}$	$\sqrt{2}$ $\{4\}_1,\{4\}_3$ $\sqrt{8}$
	$\eta=\frac{\sqrt{2}}{2}\tan\alpha$ $\frac{l_c}{l}=\cos\alpha=(1+2\eta^2)^{-1/2}$ RST $\left\{\tilde{6}(\sqrt{1/2}),4\right\}$ 90°	3 $\left\{\begin{matrix}\tilde{6}\left[\frac{\sqrt{2}}{2}\left(\frac{2\eta-1}{\eta+1}\right)\right]\\ \tilde{6}\left[\frac{\sqrt{2}}{2}\left(\frac{2\eta+1}{\eta-1}\right)\right]\end{matrix}\right\}$ 39, 4	2 $\left\{\begin{matrix}6\\4\end{matrix}\right\}_D$	4 $\left\{\begin{matrix}3\\4\end{matrix}\right\}_4$	$\left\{\begin{matrix}\tilde{4}\left[\sqrt{2}\eta/\sqrt{1+\eta^2}\right]\\ \tilde{6}\left[2\sqrt{2}\eta/\sqrt{3+\eta^2}\right]\end{matrix}\right\}$ 58	$\sqrt{2}$ $\{\infty\}$ $\sqrt{8}$
	—	See p. 76 —	$\left\{\begin{matrix}6\\ \tilde{6}[\sqrt{8}]\end{matrix}\right\}_D$ 33.6°	(exists only as a compound) —	$\left\{\begin{matrix}\tilde{6}\left[\sqrt{2}(2+\eta)/(1-\eta)\right]\\ \tilde{6}\left[\sqrt{2}\eta/(2+\eta)\right]\end{matrix}\right\}$ 85	$\sqrt{2}$ $\{3\}$ $\sqrt{2}$
	RSP $\{6,\tilde{6}(\sqrt{8})\}$ 33.6°					
P	RSP $\{4,\tilde{6}(\sqrt{2})\}_P$ 60° $\eta=\frac{\sqrt{2}}{4}\tan\alpha$ $\cos\alpha=(1+8\eta^2)^{-1/2}=\frac{l_c}{l}$	5 $\left\langle\tilde{4}\left[\frac{2\eta}{(1+4\eta^2)^{1/2}}\right],(6)\right\rangle$ 53 (pseudo-regular) or degenerate quasi-regular	6 $\left\{\begin{matrix}4\\ \tilde{6}[\sqrt{2}]\end{matrix}\right\}_P$ 60°	$\{\tilde{6}[\sqrt{2}],6\}_P$ 60°	6 $\left\{\begin{matrix}\tilde{4}\left[\sqrt{2}\eta/(1+\eta)\right]\\ \tilde{6}\left[\sqrt{2}(1+2\eta)/(1-\eta)\right]\end{matrix}\right\}$ 83, 85; $\eta=\frac{\sqrt{2}}{2}\tan\alpha$ $\cos\alpha=(1+2\eta^2)^{-1/2}=\frac{l_c}{l}$ 11	$\sqrt{2}$ $\{4\}$
	$\cos\alpha=(1+\eta^2)^{-1/2}=\frac{l_c}{l}$ $\eta=\tan\alpha$ RSP $\{6,\tilde{4}(\sqrt{2})\}$ 48.19°	7 $\left\langle\tilde{6}\left[\frac{\sqrt{2}}{\sqrt{3+\eta^2}}\eta\right],(4)\right\rangle$ 55 ~~(pseudo-regular)~~ a degenerate quasi-regular	6 $\left\{\begin{matrix}6\\ \tilde{4}[\sqrt{2}]\end{matrix}\right\}_P$ 48.19°	8 $\left\{\begin{matrix}3\\ \tilde{4}[\sqrt{2}]\end{matrix}\right\}_4$ 48.19°	~~$\tilde{4}[\sqrt{2}\eta/(1+\eta)]$, $\tilde{6}[\sqrt{2}(1+2\eta)/(1-\eta)]$~~ 87; 11 $\left\{\begin{matrix}\tilde{4}\left[\sqrt{2}(1+\eta)/(1-\eta)\right]\\ \tilde{6}\left[2\sqrt{2}\eta/(\eta+3)\right]\end{matrix}\right\}$	$\{4\}$ $\sqrt{2}$ $\sqrt{8}$
	$\eta=\frac{\sqrt{2}}{2}\tan\alpha$ $\frac{l_c}{l}=\cos\alpha=(1+2\eta^2)^{-1/2}$ RST $\{\tilde{6}(2),6\}$ 60°	9 $\left\{\begin{matrix}\tilde{6}\left[\sqrt{2}\left(\frac{\eta+1}{2\eta-1}\right)\right]\\ \tilde{6}\left[\sqrt{2}\left(\frac{\eta-1}{2\eta+1}\right)\right]\end{matrix}\right\}$ $\frac{1+\eta}{1-2\eta}$ $\frac{1-\eta}{1+2\eta}$ 41, 34, 76*, 3	$\{6,\tilde{4}[\sqrt{2}]\}_P$ 48.2°	NOTE difference from $\cancel{h}\{\tilde{4},6\}$, however. $\{6,\tilde{6}[\sqrt{8}]\}$ 33.6°	$\left\langle\tilde{6}\left[\frac{\sqrt{2}\eta}{\sqrt{3+\eta^2}}\right],(4)\right\rangle$ (pseudo-regular) 55	$\sqrt{2}$ $\{3\}_1,\{3\}_2$ $\sqrt{2}$
L	RSSP $\{\tilde{4}(\sqrt{1/2}),\tilde{6}(\sqrt{\frac{1}{2}})\}$ 70.56° 90° $\eta=\frac{\sqrt{6}}{4}\tan\alpha$ $\cos\alpha=(1+\frac{8}{3}\eta^2)^{-1/2}$	10 $\left\{\begin{matrix}\tilde{4}\left[\frac{1-2\eta}{\sqrt{2+4\eta+4\eta^2}}\right]\\ \tilde{4}\left[\frac{1+2\eta}{\sqrt{2-4\eta+4\eta^2}}\right]\end{matrix}\right\}$ 49, 3	11 $\left\{\begin{matrix}4\\ \tilde{6}[\sqrt{1/2}]\end{matrix}\right\}$ 90°	$\{\tilde{6}[\sqrt{1/2}],\tilde{6}[\sqrt{2}]\}$ 90° 60°	11 $\left\{\begin{matrix}\tilde{4}\left[\sqrt{2}\eta/\sqrt{\eta^2+2\eta+2}\right]\\ \tilde{6}\left[\sqrt{2}(2\eta-1)/\sqrt{\eta^2+2\eta+4}\right]\end{matrix}\right\}$ 44	$\sqrt{2}$ $\{4\}_1,\{4\}_3$ $\sqrt{8}$
	$\eta=\sqrt{3}\tan\alpha$ RSSP $\{\tilde{6}(\sqrt{2/7}),\tilde{4}(\sqrt{2/5})\}$ 99.593° 73.4° $\frac{l_c}{l}$ $\cos\alpha=(1+\frac{1}{3}\eta^2)^{-1/2}$	12 $\left\{\begin{matrix}\tilde{6}\left[\frac{\sqrt{2}(1-\eta)}{\sqrt{7+4\eta+\eta^2}}\right]\\ \tilde{6}\left[\frac{\sqrt{2}(1+\eta)}{\sqrt{7-4\eta+\eta^2}}\right]\end{matrix}\right\}$ 15, 2	14 $\left\{\begin{matrix}6\\ \tilde{4}[\sqrt{2/5}]\end{matrix}\right\}$ 73.4°	13 $\left\{\begin{matrix}3\\ \tilde{4}[\sqrt{2/5}]\end{matrix}\right\}_4$ 73.4°	11 $\left\{\begin{matrix}\tilde{4}\left[\sqrt{2}(1-\eta)/\sqrt{5+2\eta+\eta^2}\right]\\ \tilde{6}\left[2\sqrt{2}\eta/\sqrt{21-6\eta+\eta^2}\right]\end{matrix}\right\}$ 89; $\eta=\sqrt{5}\tan\alpha$ $\cos\alpha=(1+\frac{1}{5}\eta^2)^{-1/2}=\frac{l_c}{l}$	$\sqrt{2}$ $\{4\}_1,\{4$ $\sqrt{8}$
	$\eta=\frac{1}{2}\tan\alpha$ $\frac{l_c}{l}$ $\cos\alpha=(1+4\eta^2)^{-1/2}$ RSSP $\{\tilde{6}(\sqrt{1/2}),\tilde{6}(2)\}$ 90° 60°	14 $\left\{\begin{matrix}\tilde{6}\left[\frac{1+2\eta}{\sqrt{2(1+2\eta+4\eta^2)}}\right]\\ \tilde{6}\left[\frac{1+2\eta}{\sqrt{2(1-2\eta+4\eta^2)}}\right]\end{matrix}\right\}$ 51, 3	12 $\left\{\begin{matrix}6\\ \tilde{6}[\sqrt{2}]\end{matrix}\right\}$ 60°	15 $\left\{\begin{matrix}3\\ \tilde{6}[\sqrt{2}]\end{matrix}\right\}_6$ 60°	12 $\left\{\begin{matrix}\tilde{6}\left[\sqrt{2}(1-\eta)/\sqrt{1+\eta+\eta^2}\right]\\ \tilde{6}\left[\sqrt{2}\eta/\sqrt{3-3\eta+\eta^2}\right]\end{matrix}\right\}_2$ 91, 78	$\sqrt{2}$ $\{3\}_1,\{$ $\sqrt{2}$

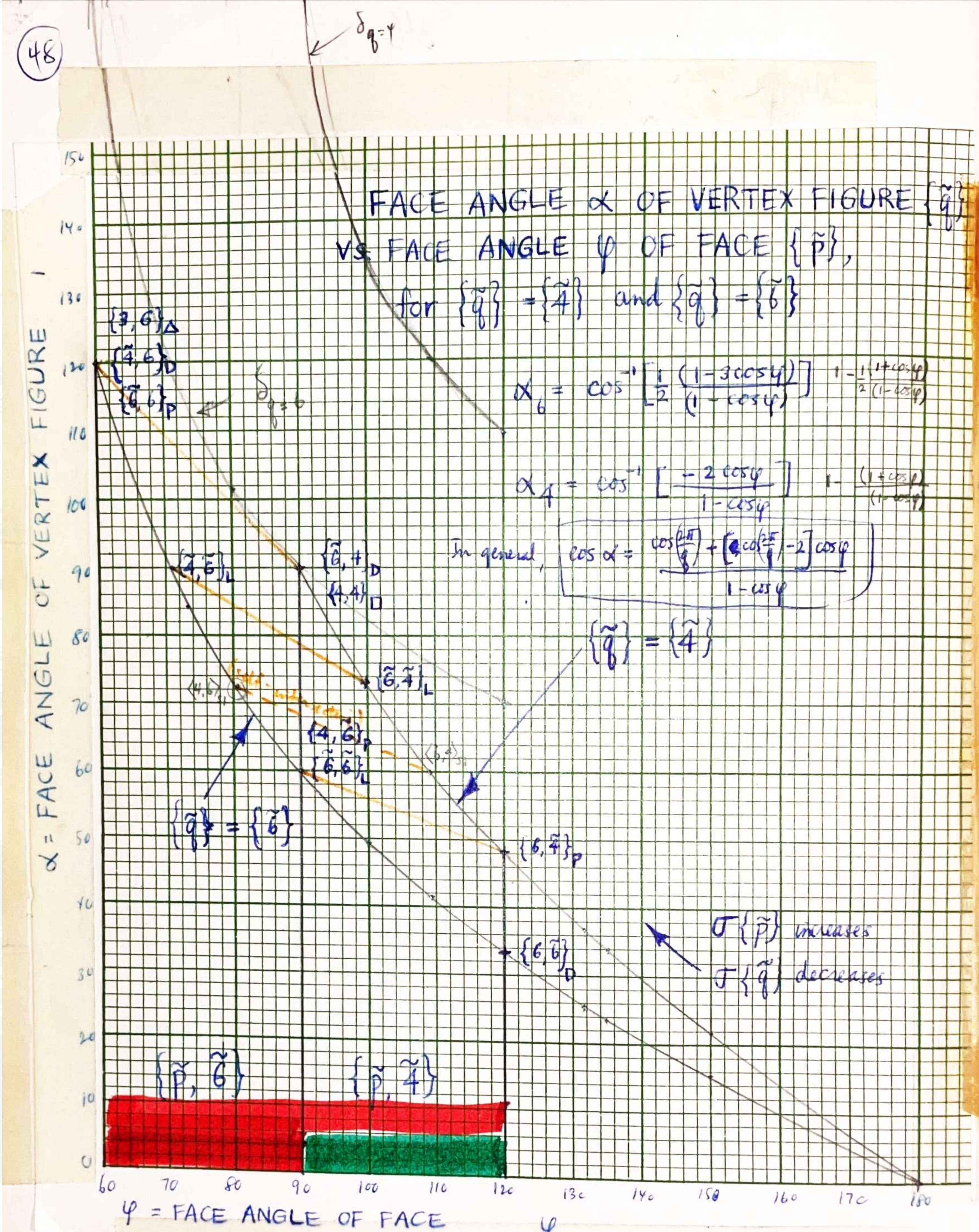
	Labyrinths	Regular figure	Quasi-regular figure (variable skewness; alternating faces)	$\mathcal{M}$	$\mathcal{N}$
REGULAR SADDLE TESSELLATIONS (∞ per. min. surfaces	$\Diamond$	$\left\{\tilde{4}[1], 6\right\}$ (60°)	$\left\{\begin{matrix}\tilde{4}\\ \tilde{4}\end{matrix}\right\}_6$ $\eta=\frac{1}{2}$ kelvin cubes (st)	$\left\{\begin{matrix}4\\ 6\end{matrix}\right\}_4$	$\left\{6, \tilde{6}[\sqrt{8}]\right\}$ (33.6°)
$\left\{\tilde{p}, q\right\}$	$\Diamond$	$\left\{\tilde{6}[\sqrt{1/2}], 4\right\}$ (90°)	$\left\{\begin{matrix}\tilde{6}(\ )\\ \tilde{6}(\ )\end{matrix}\right\}_4$ $\eta=2$ tetragonal tetrahedra (st) (IBCC)	$\left\{\begin{matrix}6\\ 4\end{matrix}\right\}_4$	$\left\{\begin{matrix}3\\ 4\end{matrix}\right\}_8$
$\left\{\begin{matrix}6\\ \tilde{6}[\sqrt{8}]\end{matrix}\right\}_6$ 33.6°	S.C.	(σ) $\left\{\tilde{6}[\sqrt{2}], 6\right\}$ (60°)	$\left\{\begin{matrix}\tilde{6}\\ \tilde{6}\end{matrix}\right\}_6$ $\eta=\frac{1}{2}$ diamond tetrahedra $\Diamond$ (st)	$\left\{6, \tilde{4}[\sqrt{2}]\right\}$ 48.2°	$\left\{6, \tilde{6}[\sqrt{8}]\right\}$ 33.6°
REGULAR SKEW POLYHEDRA	S.C.	$\left\{4, \tilde{6}[\sqrt{2}]\right\}$ (60°)	$\left\{\tilde{4}, \tilde{6}\right\}_{SC}$ tetragonal tetrahedra (st)	$\left\{\begin{matrix}4\\ \tilde{6}[\sqrt{2}]\end{matrix}\right\}_4$ (60°)	$\left\{\tilde{6}[\sqrt{2}], 6\right\}$ 60°
$\left\{p, \tilde{q}\right\}$	S.C	$\left\{6, \tilde{4}(\sqrt{2})\right\}$ 48.2° $\cos(48.2°)=\frac{2}{3}$	$\left\{\tilde{6}, \tilde{4}\right\}_{SC}$ expanded octahedra (st)	$\left\{\begin{matrix}6\\ \tilde{4}[\sqrt{2}]\end{matrix}\right\}_4$ (48.2°) canonical form $\left\{\begin{matrix}\tilde{6}[\sqrt{1/8}]\\ \tilde{4}[\sqrt{1/2}]\end{matrix}\right\}_4$ (locally centered net)	$\left\{\begin{matrix}3\\ \tilde{4}[\sqrt{2}]\end{matrix}\right\}_8$ 48.2°
	$\Diamond$	$\left\{6, \tilde{6}[\sqrt{8}]\right\}$ 33.6° $\cos^{-1}\left(\frac{5}{6}\right)$	~~$\left\{\tilde{6}, \tilde{6}\right\}$~~	$\left\{\begin{matrix}6\\ \tilde{6}[\sqrt{8}]\end{matrix}\right\}_4$ 33.6° $\xrightarrow[(\eta=2)]{(\sigma)}$ tetragonal tetrahedra	(doesn't exist) [except as a compound]
REGULAR SKEW SADDLE POLYHEDRA	Laves	$\left\{\tilde{4}[\sqrt{1/2}], \tilde{6}[\sqrt{1/2}]\right\}$ 70.53° 90° $\cos^{-1}\left(\frac{1}{3}\right)$	$\left\{\begin{matrix}\tilde{4}\\ \tilde{4}\end{matrix}\right\}_6$ "lim" $\left\{\begin{matrix}4\\ \tilde{7}[29.92 \ldots]\end{matrix}\right\}$	$\left\{\begin{matrix}4\\ \tilde{6}[\sqrt{1/2}]\end{matrix}\right\}_4$ 90° σ	$\left\{\tilde{6}[\sqrt{1/2}], \tilde{6}[\sqrt{2}]\right\}$ 90° 60°
$\left\{\tilde{p}, \tilde{q}\right\}$	Laves	$\left\{\tilde{6}[\sqrt{2/7}], \tilde{4}[\sqrt{2/5}]\right\}$ 99.953° 73.4°	$\left\{\begin{matrix}\tilde{6}\\ \tilde{6}\end{matrix}\right\}_4$	$\left\{\begin{matrix}6\\ \tilde{4}[\sqrt{2/5}]\end{matrix}\right\}_4$ 73.4°	$\left\{\begin{matrix}3\\ \tilde{4}[\sqrt{2/5}]\end{matrix}\right\}_8$ 73.4°
	Laves	$\left\{\tilde{6}[\sqrt{1/2}], \tilde{6}[\sqrt{2}]\right\}$ 90° 60°	$\left\{\begin{matrix}\tilde{6}\\ \tilde{6}\end{matrix}\right\}_6$ lim $\left\{\begin{matrix}6\\ [illegible]\end{matrix}\right\}_6$	σ $\left\{\begin{matrix}6\\ \tilde{6}[\sqrt{2}]\end{matrix}\right\}_4$ 60°	$\left\{\begin{matrix}3\\ \tilde{6}[\sqrt{2}]\end{matrix}\right\}_{12}$ 60°

SCHWARZ

COXETER

LAVES

(48)



$\delta_{q=4}$
FACE ANGLE α OF VERTEX FIGURE $\{\tilde{q}\}$
vs FACE ANGLE φ OF FACE $\{\tilde{p}\}$,
for $\{\tilde{q}\} = \{\tilde{4}\}$ and $\{\tilde{q}\} = \{\tilde{6}\}$
$\alpha_6 = \cos^{-1}\left[\frac{1}{2}\frac{(1-3\cos\varphi)}{(1-\cos\varphi)}\right]$
$1 - \frac{1}{2}\frac{(1+\cos\varphi)}{(1-\cos\varphi)}$
$\alpha_4 = \cos^{-1}\left[\frac{-2\cos\varphi}{1-\cos\varphi}\right]$
$1 - \frac{(1+\cos\varphi)}{(1-\cos\varphi)}$
In general,
$\cos\alpha = \frac{\cos\left(\frac{2\pi}{q}\right) + \left[\cos\left(\frac{2\pi}{q}\right) - 2\right]\cos\varphi}{1-\cos\varphi}$
$\{3,6\}_\Delta$
$\{\tilde{4},6\}_D$
$\{\tilde{6},6\}_P$
$\delta_{q=6}$
$\{\tilde{4},\tilde{6}\}_L$
$\{\tilde{6},4\}_D$
$\{4,4\}_\square$
$\{\tilde{q}\} = \{\tilde{4}\}$
$\{4,6\}_H$
$\{\tilde{6},\tilde{4}\}_L$
$\{4,\tilde{6}\}_P$
$\{\tilde{6},\tilde{6}\}_L$
$\{6,4\}_H$
$\{\tilde{q}\} = \{\tilde{6}\}$
$\{6,\tilde{4}\}_P$
$\{6,\tilde{6}\}_D$
$\sigma\{\tilde{p}\}$ increases
$\sigma\{\tilde{q}\}$ decreases
$\{\tilde{p},\tilde{6}\}$
$\{\tilde{p},\tilde{4}\}$
α = FACE ANGLE OF VERTEX FIGURE
φ = FACE ANGLE OF FACE
φ
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
60
70
80
90
100
110
120
130
140
150
160
170
180

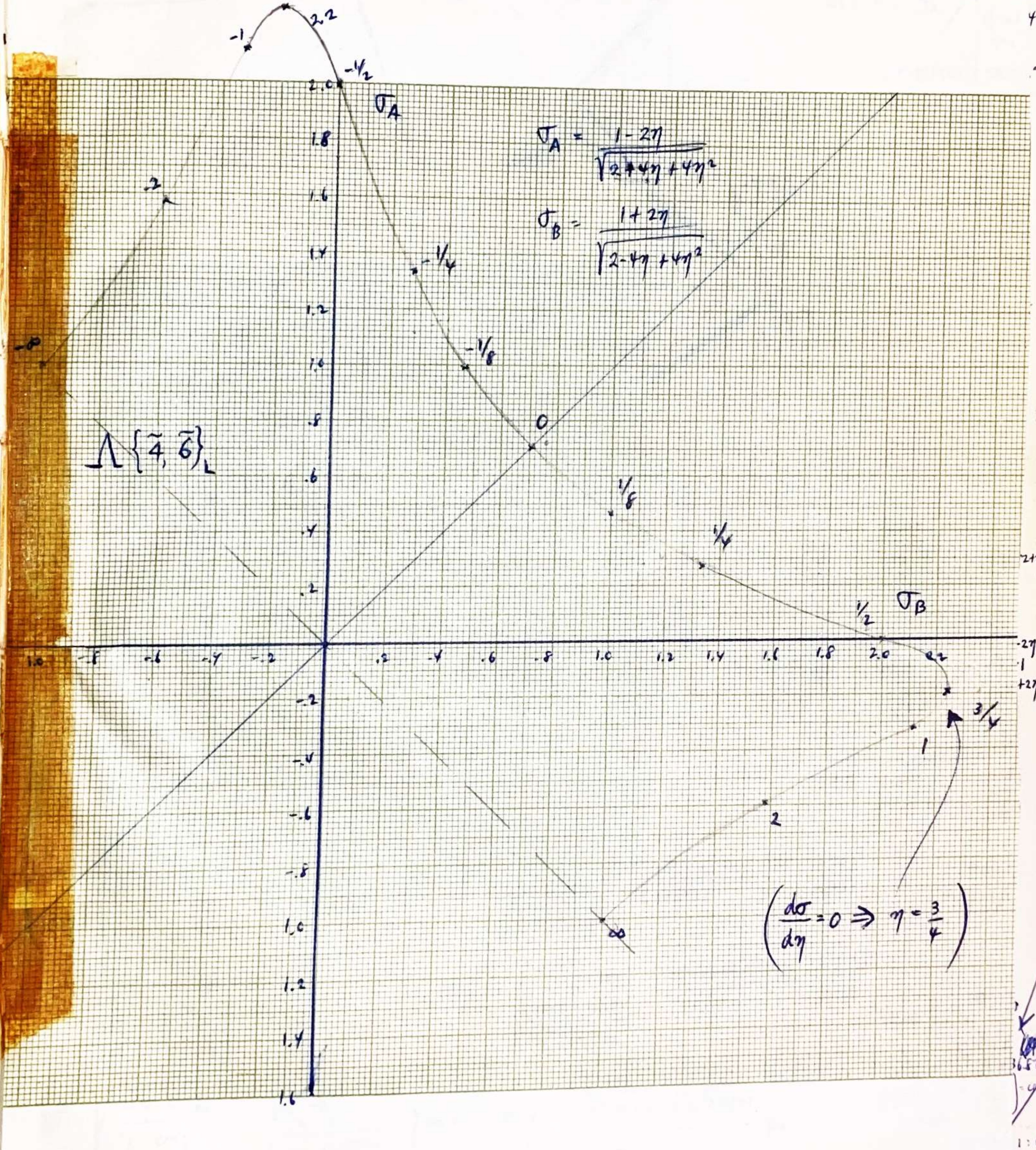
Alternate vertices are displaced with opposite lat[illegible] here.

$$\boxed{\sigma\{\tilde{4},\tilde{6}\}_L \rightarrow \left\{\begin{matrix}\tilde{4}\\ \tilde{4}\end{matrix}\right\}_L}$$

If $\sigma_a = 0$ $\Big\}$ $\varphi_a = 90°$

in $21.8?° = \cos^{-1}(.8)$

435



$36.87°$
$1:1300$!!)

$\sigma_{11}^{(B)} = \Big|\Big/ \dfrac{(1+2\eta)^2}{\ldots}$ $\Big|\dfrac{(1+2\eta)^2}{\ldots}\Big|^{1/2}$ $\Big\{\dfrac{(1+2\eta)}{\sqrt{\ldots}}\Big\} = \Big|\dfrac{1}{2}$ ✓

have no real roots.
The array of displacements does not bring fundamental

$\therefore$ $\boxed{\varphi = 29.926434°}$

(by computer!)

σ_2 (same signs)

$$(1-2\eta)^2+(2\eta)^2 = 1-4\eta+4\eta^2+4\eta^2 = 1-4\eta+8\eta^2$$

$$(1+2\eta)^2+(2\eta)^2 = 1+4\eta+4\eta^2+4\eta^2 = 1+4\eta+8\eta^2$$

Let $l_1=0$

$8\eta^2-4\eta+1=0$

$\eta=\frac{4\pm\sqrt{16-32}}{16}$

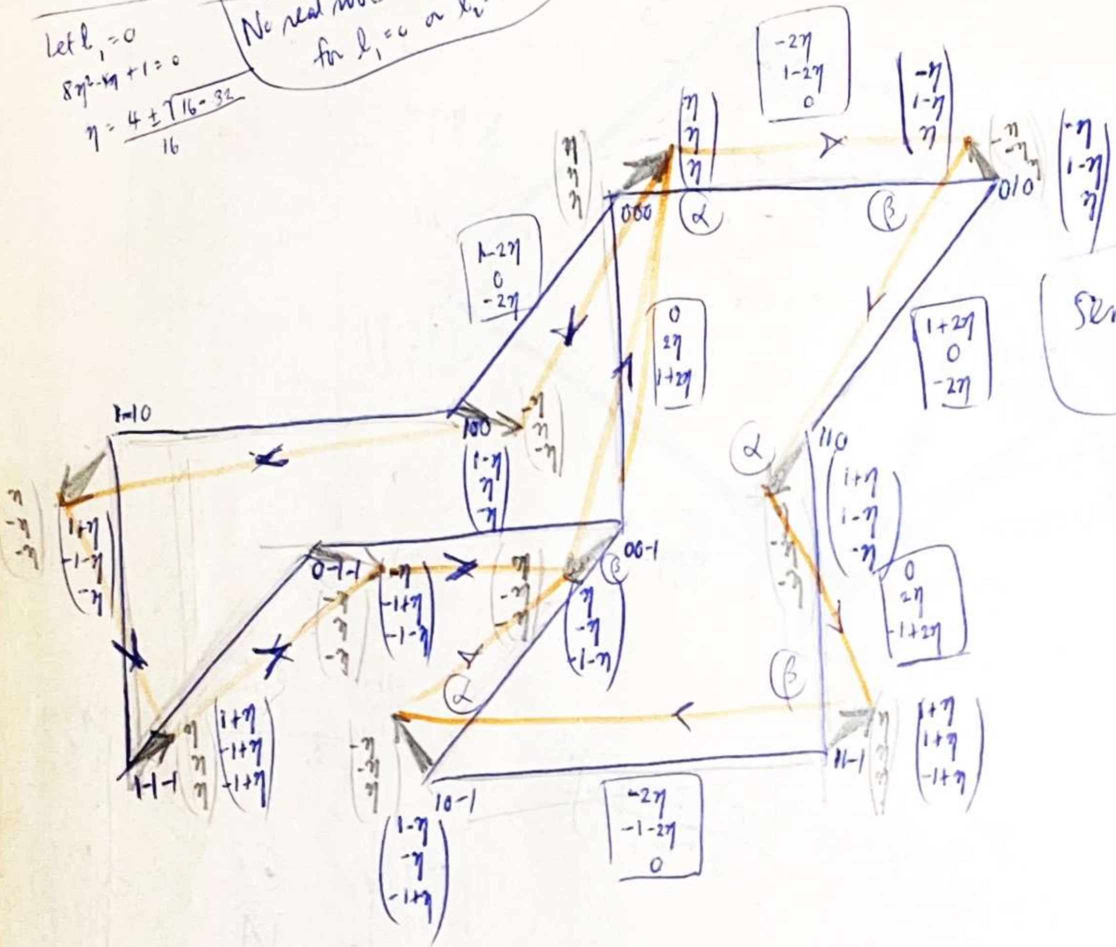
No real roots for $l_1=0$ or $l_2=0$

$$\cos\varphi_\alpha=\frac{\begin{pmatrix}-2\eta\\1-2\eta\\0\end{pmatrix}\begin{pmatrix}0\\-2\eta\\-1-2\eta\end{pmatrix}}{\underbrace{\sqrt{1-4\eta+8\eta^2}}_{(l_1)}\underbrace{\sqrt{1+4\eta+8\eta^2}}_{(l_2)}}=\frac{-2\eta+4\eta^2=2\eta(2\eta-1)}{\sqrt{\ }\sqrt{\ }}\quad\boxed{\frac{2\eta(2\eta-1)}{\sqrt{1-4\eta+8\eta^2}\sqrt{1+4\eta+8\eta^2}}}$$

$$\cos\varphi_\beta=\frac{\begin{pmatrix}2\eta\\-1+2\eta\\0\end{pmatrix}\begin{pmatrix}1+2\eta\\0\\-2\eta\end{pmatrix}}{l_1 l_2}=\frac{2\eta(1+2\eta)}{l_1 l_2}=\boxed{\frac{2\eta(2\eta+1)}{\sqrt{1-4\eta+8\eta^2}\sqrt{1+4\eta+8\eta^2}}}$$

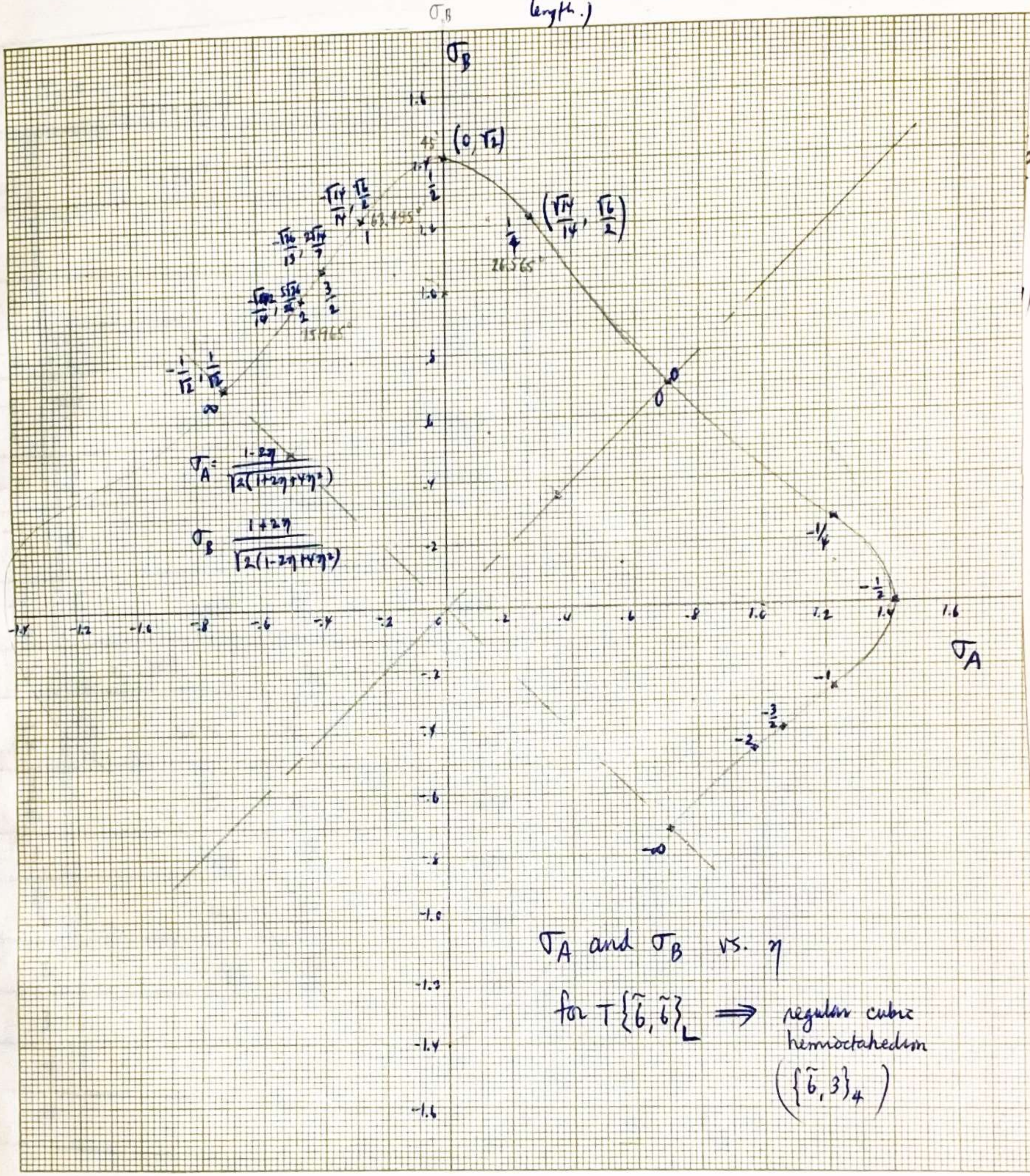
$\varphi_\alpha\uparrow(\eta)$

$\varphi_\beta\downarrow(\eta)$



Semi-regular polyhedron

Vertex collision occurs at $\eta = \frac{1}{2}$ (or $\eta = -\frac{1}{2}$). (If all vertices are displaced from one labyrinth into the other, the edges do not remain equal in length.)



Variable skewness quasi-regular $\left\{ \begin{matrix} 6 \\ \tilde{6} \end{matrix} \right\}_6$

i.e., 60° hexes and plane hexes can be joined either by fours or sixes to make a quasi-regular polyhedron!

η	$\sigma_A = \frac{1-2\eta}{\sqrt{2(1+2\eta+4\eta^2)}}$	$\sigma_B = \frac{1+2\eta}{\sqrt{2(1-2\eta+4\eta^2)}}$
$-\infty$	.707	-.707
-4		
-3		
-2	$\frac{5\sqrt{26}}{26}$ .981	-.463
$-\frac{3}{2}$	$\frac{2\sqrt{14}}{7}$ 1.069	-.3923
-1	$\frac{\sqrt{6}}{2}$ 1.224	-.2673
$-\frac{1}{2}$	$\sqrt{2}$ 1.414	0
$-\frac{1}{4}$	$\frac{\sqrt{6}}{2}$ 1.224	$\frac{\sqrt{14}}{14}$ = -.2676
0	$\frac{1}{\sqrt{2}}$ = .707	$\frac{1}{\sqrt{2}}$ = .707
$\frac{1}{4}$	$\frac{\sqrt{14}}{14}$ = .2676	$\frac{\sqrt{6}}{2}$ 1.224
$\frac{1}{2}$	0	$\sqrt{2}$ 1.414
1	-.2673	$\frac{\sqrt{6}}{2}$ 1.224
$3/2$	-.3923	$\frac{2\sqrt{14}}{7}$ 1.069
2	-.463	$\frac{5\sqrt{26}}{26}$ .981
3		
4		
∞		.707